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Subleading-color Contributions to Gluon-gluon Scattering in $\mathcal{N}=4$ SYM Theory and Relations to $\mathcal{N}=8$ Supergravity

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Abstract

We study the subleading-color (nonplanar) contributions to the four-gluon scattering amplitudes in $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory. Using the formalisms of Catani and of Sterman and Tejeda-Yeomans, we develop explicit expressions for the infrared-divergent contributions of all the subleading-color L-loop amplitudes up to three loops, and make some conjectures for the IR behavior for arbitrary L. We also derive several intriguing relations between the subleading-color one- and two-loop four-gluon amplitudes and the four-graviton amplitudes of $\mathcal{N}=8$ supergravity. The exact one- and two-loop $\mathcal{N}=8$ supergravity amplitudes can be expressed in terms of the one- and two-loop N-independent $\mathcal{N}=4$ SYM amplitudes respectively, but the natural generalization to higher loops fails, despite having a simple interpretation in terms of the 't Hooft picture. We also find that, at least through two loops, the subleading-color amplitudes of $\mathcal{N}=4$ SYM theory have uniform transcendentality (as do the leading-color amplitudes). Moreover, the $\mathcal{N}=4$ SYM Catani operators, which express the IR-divergent contributions of loop amplitudes in terms of lower-loop amplitudes, are also shown to have uniform transcendentality, and to be the maximum transcendentality piece of the QCD Catani operators.

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1 Introduction

In the effort to develop new tools for the computation of higher-loop contributions to scattering amplitudes in gauge theories, $\mathcal{N}=4$ SYM theory plays a special role because of its comparatively simple structure [1]. Moreover, the AdS/CFT correspondence allows some of the observables of this theory to be computed in the strong coupling limit [2–4].

The two-loop four-gluon scattering amplitude was first computed for $\mathcal{N}=4$ SU(N) SYM theory by Bern, Rosowsky, and Yan [5] using cutting techniques, with the results expressed in terms of two-loop planar and non-planar scalar integrals (see also ref. [6]). Explicit expressions for these IR-divergent integrals as Laurent expansions in ϵ (where $D=4-2\epsilon$) were later obtained by Smirnov in the planar case [7], and by Tausk in the non-planar case [8]. Subsequently, Anastasiou, Bern, Dixon, and Kosower (ABDK) demonstrated that the two-loop amplitude is expressible in terms of the one-loop amplitude in the large-N (leading color) limit, suggesting an iterative structure for the loop expansion of $\mathcal{N}=4$ SYM amplitudes in this same limit [1]. Using insights from decades of study of the IR divergences of gauge theory amplitudes [9–19], Bern, Dixon, and Smirnov conjectured a complete nonperturbative exponential ansatz for MHV n-gluon scattering amplitudes [20], again in the leading-color (large-N) limit. In this limit, only planar diagrams (in the topological expansion of 't Hooft) contribute [21].

In this paper, we explore the structure of the subleading-color (nonplanar) contributions to the four-gluon amplitude in $\mathcal{N}=4$ SU(N) SYM theory through three loops, focusing particularly on the IR-divergent terms. We also demonstrate some intriguing connections between these subleading-color amplitudes and four-graviton amplitudes in $\mathcal{N}=8$ supergravity.

After reviewing the known exact one- and two-loop results for the full four-gluon amplitude in $\mathcal{N}=4$ SYM theory, we use Catani's formalism [18] to develop explicit expressions for the IR-divergent terms of the one- and two-loop subleading-color amplitudes, and a combination of his approach and that of Sterman and Tejeda-Yeomans [19] for three loops. We denote the leading-color L-loop amplitude by $A^{(L,0)}$, and the subleading-color amplitudes by $A^{(L,k)}$, $(k=1,\cdots,L)$ where $A^{(L,k)}$ is the component of the amplitude proportional to N^{L-k}. We show (for $L\leq 3$) that the leading IR divergence of the amplitude $A^{(L,k)}$ is $\mathcal{O}(1/\epsilon^{2L-k})$, and explicitly determine its coefficient. We find that the first two terms of the Laurent expansion of the N-independent amplitude $A^{(L,L)}$ (which starts at $\mathcal{O}(1/\epsilon^L)$) obey the relationship

$$A^{(L,L)}(\epsilon) \quad \sim \quad \frac{P_{L-1}(X,Y,Z)}{\epsilon^{L-1}} A^{(1,1)}(L\epsilon) + \mathcal{O}\left(\frac{1}{\epsilon^{L-2}}\right) \tag{1.1}$$

where $P_n(X,Y,Z)$ is an nth order polynomial, explicitly specified in eqs. (4.45) and (4.46), and $X=\log(t/u)$, $Y=\log(u/s)$, and $Z=\log(s/t)$, with s, t, and u being Mandelstam parameters. We prove eq. (1.1) for two and three loops (cf. eqs. (4.26) and (4.34)), and conjecture the result to be valid generally.

Next, observing that the N-independent SYM amplitude $A^{(L,L)}$ has a leading divergence of the same order as the L-loop four-graviton $\mathcal{N}=8$ supergravity amplitude [22, 23], we show in sec. 5 that the full amplitudes are related. In particular, we demonstrate the exact

relationships for one- and two-loop amplitudes

$$\frac{1}{3}\left[(\lambda_{\rm SG} u) M^{(1,1)}_{\rm SYM}(s,t) + \text{cyclic permutations}\right] = \sqrt{2}\lambda_{\rm SYM} M^{(1)}_{\rm SG} \tag{1.2}$$

$$\frac{1}{3}\left[(\lambda_{\rm SG} u)^2 M^{(2,2)}_{\rm SYM}(s,t) + \text{cyclic permutations}\right] = (\sqrt{2}\lambda_{\rm SYM})^2 M^{(2)}_{\rm SG} \tag{1.3}$$

where $M^{(L,L)}_{\rm SYM}$ and $M^{(L)}_{\rm SG}$ are ratios of L-loop N-independent SYM amplitudes $A^{(L,L)}_{\rm SYM}$ and L-loop supergravity amplitudes $A^{(L)}_{\rm SG}$ to tree-level amplitudes, and $\lambda_{\rm SYM} = g^2 N$ and $\lambda_{\rm SG} = (\kappa/2)^2$ are SYM and supergravity coupling constants. The natural generalization of eqs. (1.2) and (1.3) to L loops is not satisfied, at least in its simplest form. The one- and two-loop relations have a simple (albeit nonintuitive) interpretation in terms of the 't Hooft picture, once we factor in an unusual identification, in that the topological expansion of the SYM Feynman diagrams is related to the loop expansion of supergravity.

In sec. 6, we discuss the transcendentality of the $\mathcal{N}=4$ SYM amplitudes as well as that of the Catani *operators*, which has not previously appeared in the literature. We observe that the subleading-color $\mathcal{N}=4$ SYM amplitudes through two loops have uniform transcendentality, as is already known for the leading-color amplitudes. Moreover, the $\mathcal{N}=4$ SYM Catani operators (at least through three loops) also have uniform transcendentality, which implies the same for the divergent contributions of n-gluon scattering amplitudes through that loop order. Finally, the $\mathcal{N}=4$ SYM Catani operators constitute the maximum transcendentality piece of the corresponding QCD Catani operators.

2 One- and two-loop four-gluon amplitudes

In this section, we review known exact results for one- and two-loop four-gluon amplitudes in $\mathcal{N}=4$ SU(N) SYM theory, and relations among them.

Gluon n-point amplitudes may be written in a color-decomposed form as a sum over single and multiple traces of color generators [24]. The four-gluon amplitude contains only single and double traces, and can be written as

$$\begin{aligned}\mathcal{A}_{4-\rm gluon}(1,2,3,4) &= \sum_{\sigma\in S_4/\mathsf{Z}_4} {\rm Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\ &+ \sum_{\sigma\in S_4/\mathsf{Z}_2^2} {\rm Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}){\rm Tr}(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) A_{4;3}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))\end{aligned} \tag{2.1}$$

where the color-stripped amplitudes $A_{4;1}$ and $A_{4;3}$ implicitly depend on the momenta and polarizations of the external particles, and where T^a are SU(N) generators in the fundamental representation, normalized according to ${\rm Tr}(T^aT^b)=\delta^{ab}$. The L-loop diagrams contributing to $A_{4;1}$ start at order N^L, those contributing to $A_{4;3}$ start at order N^{L-1}, and corrections to each term come with factors of $1/N^2$ (from loop index traces), so that the L-loop amplitude has the form

$$\begin{aligned} A_{4;1} &= g^2 a^L \left[A^{(L,0)}_{4;1} + \frac{1}{N^2}A^{(L,2)}_{4;1} + \cdots\right] \\ A_{4;3} &= g^2 a^L \left[\frac{1}{N}A^{(L,1)}_{4;3} + \frac{1}{N^3}A^{(L,3)}_{4;3} + \cdots\right]\end{aligned} \tag{2.2}$$

with the series ending at the N-independent amplitude $A^{(L,L)}$, and where the natural 't Hooft loop expansion parameter is [20]

$$a \equiv \frac{g^2 N}{8\pi^2}\left(4\pi \mathrm{e}^{-\gamma}\right)^{\epsilon} . \tag{2.3}$$

Here γ is Euler's constant, and the loop amplitudes, being IR-divergent, are evaluated using dimensional regularization in $D = 4 - 2\epsilon$ dimensions. The leading-color term $A^{(L,0)}$ comes from planar diagrams, whereas the subleading-color terms $A^{(L,1)}$ through $A^{(L,L)}$ include contributions from nonplanar diagrams.

We will also find it convenient to write the four-gluon amplitude as

$$\begin{aligned}
\mathcal{A}_{4-\mathrm{gluon}}(1,2,3,4) &= g^2 \sum_{L=0}^{\infty} a^L \sum_{i=1}^{9} A_{[i]}^{(L)}\, \mathcal{C}_{[i]} \\
&= g^2 \sum_{L=0}^{\infty} a^L \sum_{k=0}^{L} \frac{1}{N^k} \sum_{i=1}^{9} A_{[i]}^{(L,k)}\, \mathcal{C}_{[i]}
\end{aligned} \tag{2.4}$$

in terms of an explicit basis of traces [25]

$$\begin{array}{lll}
\mathcal{C}_{[1]} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,, & \mathcal{C}_{[4]} = \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})\,, & \mathcal{C}_{[7]} = \mathrm{Tr}(T^{a_1}T^{a_2})\mathrm{Tr}(T^{a_3}T^{a_4}) \\
\mathcal{C}_{[2]} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\,, & \mathcal{C}_{[5]} = \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})\,, & \mathcal{C}_{[8]} = \mathrm{Tr}(T^{a_1}T^{a_3})\mathrm{Tr}(T^{a_2}T^{a_4}) \\
\mathcal{C}_{[3]} = \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\,, & \mathcal{C}_{[6]} = \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})\,, & \mathcal{C}_{[9]} = \mathrm{Tr}(T^{a_1}T^{a_4})\mathrm{Tr}(T^{a_2}T^{a_3})
\end{array} \tag{2.5}$$

so that $A_{[1]}^{(L,2k)}$ through $A_{[6]}^{(L,2k)}$ correspond to the single-trace amplitudes $A_{4;1}^{(L,2k)}$, and $A_{[7]}^{(L,2k+1)}$ through $A_{[9]}^{(L,2k+1)}$ correspond to the double-trace amplitudes $A_{4;3}^{(L,2k+1)}$.

The tree-level amplitudes are

$$\left(A_{[1]}^{(0)},\, A_{[2]}^{(0)},\, A_{[3]}^{(0)},\, A_{[4]}^{(0)},\, A_{[5]}^{(0)},\, A_{[6]}^{(0)}\right) = -\frac{4iK}{stu}\,(u,t,s,s,t,u) \tag{2.6}$$

where $s = (k_1+k_2)^2$, $t = (k_1+k_4)^2$, and $u = (k_1+k_3)^2$ are the usual Mandelstam variables, obeying $s+t+u=0$ for massless external gluons. The factor K, defined in eq. (7.4.42) of ref. [26], depends on the momenta and helicity of the external gluons, and is totally symmetric under permutations of the external legs. The identities $A_{[1]}^{(L)} = A_{[6]}^{(L)}$, $A_{[2]}^{(L)} = A_{[5]}^{(L)}$, and $A_{[3]}^{(L)} = A_{[4]}^{(L)}$ are satisfied at all loop orders.

At one loop, the single-trace amplitudes are given by [27]

$$A_{[1]}^{(1,0)} = M^{(1)}(s,t)\, A_{[1]}^{(0)} = 2iK\, I_4^{(1)}(s,t) \tag{2.7}$$

with the other single-trace amplitudes $A_{[2]}^{(1,0)}$ and $A_{[3]}^{(1,0)}$ obtained by letting $t \leftrightarrow u$ and $s \leftrightarrow u$ respectively. In eq. (2.7), $I_4^{(1)}(s,t)$ denotes the scalar box integral

$$\begin{aligned}
M^{(1)}(s,t) &= -\frac{1}{2} st\, I_4^{(1)}(s,t) \\
I_4^{(1)}(s,t) = I_4^{(1)}(t,s) &= -i\mu^{2\epsilon}\mathrm{e}^{\epsilon\gamma}\pi^{-D/2}\int \frac{d^D p}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}
\end{aligned} \tag{2.8}$$

an explicit expression for which is given, e.g., in ref. [20]. The one-loop double-trace amplitudes are given by [27]

$$A_{[7]}^{(1,1)} = A_{[8]}^{(1,1)} = A_{[9]}^{(1,1)} = 2\left(A_{[1]}^{(1,0)} + A_{[2]}^{(1,0)} + A_{[3]}^{(1,0)}\right) \tag{2.9}$$

$$= 4iK\left[I_4^{(1)}(s,t) + I_4^{(1)}(t,u) + I_4^{(1)}(u,s)\right]. \tag{2.10}$$

The relation (2.9) follows from the one-loop U(1) decoupling identity [24].

At two loops, the leading-color single-trace amplitude is given by [5]

$$A_{[1]}^{(2,0)} = M^{(2)}(s,t)\, A_{[1]}^{(0)} = -iK\left[sI_4^{(2)P}(s,t) + tI_4^{(2)P}(t,s)\right] \tag{2.11}$$

where $I_4^{(2)P}(s,t)$ denotes the scalar double-box (planar) integral

$$\begin{aligned} M^{(2)}(s,t) &= \frac{1}{4}st\left[sI_4^{(2)P}(s,t) + tI_4^{(2)P}(t,s)\right] \\ I_4^{(2)P}(s,t) &= \left(-i\mu^{2\epsilon}\mathrm{e}^{\epsilon\gamma}\pi^{-D/2}\right)^2 \int \frac{d^Dp\, d^Dq}{p^2\,(p+q)^2 q^2\,(p-k_1)^2\,(p-k_1-k_2)^2\,(q-k_4)^2\,(q-k_3-k_4)^2} \end{aligned} \tag{2.12}$$

an explicit expression for which is given, e.g., in ref. [20]. The double-trace amplitude is [5]

$$\begin{aligned} A_{[7]}^{(2,1)} = -2iK\Big[&s\left(3I_4^{(2)P}(s,t) + 2I_4^{(2)NP}(s,t) + 3I_4^{(2)P}(s,u) + 2I_4^{(2)NP}(s,u)\right) \\ &-t\left(I_4^{(2)NP}(t,s) + I_4^{(2)NP}(t,u)\right) - u\left(I_4^{(2)NP}(u,s) + I_4^{(2)NP}(u,t)\right)\Big] \end{aligned} \tag{2.13}$$

and the subleading-color single-trace amplitude is [5]

$$\begin{aligned} A_{[1]}^{(2,2)} = -2iK\Big[&s\left(I_4^{(2)P}(s,t) + I_4^{(2)NP}(s,t) + I_4^{(2)P}(s,u) + I_4^{(2)NP}(s,u)\right) \\ &+t\left(I_4^{(2)P}(t,s) + I_4^{(2)NP}(t,s) + I_4^{(2)P}(t,u) + I_4^{(2)NP}(t,u)\right) \\ &-2u\left(I_4^{(2)P}(u,s) + I_4^{(2)NP}(u,s) + I_4^{(2)P}(u,t) + I_4^{(2)NP}(u,t)\right)\Big] \end{aligned} \tag{2.14}$$

where $I_4^{(2)NP}(s,t)$ denotes the two-loop non-planar integral

$$I_4^{(2)NP}(s,t) = \left(-i\mu^{2\epsilon}\mathrm{e}^{\epsilon\gamma}\pi^{-D/2}\right)^2 \int \frac{d^Dp\, d^Dq}{p^2\,(p+q)^2\,q^2\,(p-k_2)^2\,(p+q+k_1)^2\,(q-k_3)^2\,(q-k_3-k_4)^2} \tag{2.15}$$

an explicit expression for which is given in ref. [8]. All the other single- and double-trace amplitudes $A_{[i]}^{(2,k)}$ are obtained by making the appropriate permutations of s, t, and u in these expressions.

The two-loop amplitudes obey the following group theory relations [28]

$$\begin{aligned} A_{[7]}^{(2,1)} &= 2\left(A_{[1]}^{(2,0)} + A_{[2]}^{(2,0)} + A_{[3]}^{(2,0)}\right) - A_{[3]}^{(2,2)} \\ A_{[8]}^{(2,1)} &= 2\left(A_{[1]}^{(2,0)} + A_{[2]}^{(2,0)} + A_{[3]}^{(2,0)}\right) - A_{[1]}^{(2,2)} \\ A_{[9]}^{(2,1)} &= 2\left(A_{[1]}^{(2,0)} + A_{[2]}^{(2,0)} + A_{[3]}^{(2,0)}\right) - A_{[2]}^{(2,2)} \end{aligned} \tag{2.16}$$

and may be easily verified using eqs. (2.11), (2.13), and (2.14). In addition, we have

$$A_{[1]}^{(2,2)} + A_{[2]}^{(2,2)} + A_{[3]}^{(2,2)} = 0 \tag{2.17}$$

also easily verified using eq. (2.14). Together these equations imply

$$6 \sum_{i=1}^{3} A_{[i]}^{(2,0)} - \sum_{i=7}^{9} A_{[i]}^{(2,1)} = 0 \tag{2.18}$$

which is the two-loop generalization of the U(1) decoupling relation (2.9). Both eqs. (2.17) and (2.18) are encapsulated in the equation

$$6 \sum_{i=1}^{3} A_{[i]}^{(L)} - N \sum_{i=7}^{9} A_{[i]}^{(L)} = 0, \qquad L \leq 2 \tag{2.19}$$

which is valid through two loops.

3 IR divergences of $\mathcal{N} = 4$ SYM amplitudes

When we dimensionally regularize a theory in $D = 4 - 2\epsilon$ dimensions, both UV and IR divergences appear as poles in ϵ. In a UV finite theory, such as $\mathcal{N} = 4$ SYM, the poles in ϵ are solely due to IR divergences. In gluon-gluon scattering in $\mathcal{N} = 4$ SYM, IR divergences arise both from soft gluons and from collinear gluons (which can exchange a virtual gluon with soft transverse momentum), each of which gives rise to an $\mathcal{O}(1/\epsilon)$ pole at 1-loop, leading to an $\mathcal{O}(1/\epsilon^2)$ pole at that order. At L loops, the leading IR divergence of $A^{(L,0)}$ is therefore $\mathcal{O}(1/\epsilon^{2L})$, arising from multiple soft gluon exchanges. The IR divergences of subleading-color amplitudes $A^{(L,k)}$, however, are not so severe, being only $\mathcal{O}(1/\epsilon^{2L-k})$. In particular, the N-independent amplitude $A^{(L,L)}$ has a leading $\mathcal{O}(1/\epsilon^{L})$ divergence, the same degree of IR divergence as an L-loop $\mathcal{N} = 8$ supergravity amplitude. As we will see in sec. 5, there are some intriguing connections between the N-independent $\mathcal{N} = 4$ SYM amplitude $A^{(L,L)}$ and the L-loop $\mathcal{N} = 8$ supergravity amplitude.

In this section and the next, we will analyze the IR-divergent contributions of leading- and subleading-color $\mathcal{N} = 4$ SYM amplitudes up to three loops using the general analysis of refs. [18, 19]. This will allow us to illustrate the IR behavior described above as well as to derive some relations for subleading-color amplitudes. In this section, it will be useful to organize the color-stripped amplitudes $A_{[i]}^{(L)}$ defined in eq. (2.4) into a vector

$$|A^{(L)}\rangle = \left(A_{[1]}^{(L)}, A_{[2]}^{(L)}, A_{[3]}^{(L)}, A_{[4]}^{(L)}, A_{[5]}^{(L)}, A_{[6]}^{(L)}, A_{[7]}^{(L)}, A_{[8]}^{(L)}, A_{[9]}^{(L)} \right)^T \tag{3.1}$$

where $(\cdots)^T$ denotes the transposed vector, so that the loop expansion of the four-gluon amplitude (2.4) may be expressed as

$$\mathcal{A}_{4-\text{gluon}} = g^2 \Big[|A^{(0)}\rangle + a|A^{(1)}\rangle + a^2|A^{(2)}\rangle + a^3|A^{(3)}\rangle + \cdots \Big], \qquad a \equiv \frac{g^2 N}{8\pi^2} \left(4\pi \mathrm{e}^{-\gamma}\right)^{\epsilon}. \tag{3.2}$$

3.1 Catani's Formalism

In ref. [18], Catani showed that the IR divergences of the L-loop amplitude $|A^{(L)}(\epsilon)\rangle$ can be characterized in terms of operators $\mathbf{I}^{(L)}$ acting on lower-order terms in the loop expansion

$$|A^{(1)}(\epsilon)\rangle = \frac{1}{N}\mathbf{I}^{(1)}(\epsilon)|A^{(0)}\rangle + |A^{(1f)}(\epsilon)\rangle \tag{3.3}$$

$$|A^{(2)}(\epsilon)\rangle = \frac{1}{N^2}\mathbf{I}^{(2)}(\epsilon)|A^{(0)}\rangle + \frac{1}{N}\mathbf{I}^{(1)}(\epsilon)|A^{(1)}(\epsilon)\rangle + |A^{(2f)}(\epsilon)\rangle \tag{3.4}$$

$$|A^{(3)}(\epsilon)\rangle = \frac{1}{N^3}\mathbf{I}^{(3)}(\epsilon)|A^{(0)}\rangle + \frac{1}{N^2}\mathbf{I}^{(2)}(\epsilon)|A^{(1)}\rangle + \frac{1}{N}\mathbf{I}^{(1)}(\epsilon)|A^{(2)}(\epsilon)\rangle + |A^{(3f)}(\epsilon)\rangle \tag{3.5}$$

where $\mathbf{I}^{(L)}(\epsilon)$ contains the terms that diverge as $\epsilon \to 0$, and $|A^{(Lf)}(\epsilon)\rangle$ is finite as $\epsilon \to 0$ (but is not the entire finite piece of $|A^{(L)}(\epsilon)\rangle$, since $\mathbf{I}^{(L)}(\epsilon)$ contains finite terms as well).

If we specialize to four-gluon scattering in $\mathcal{N}=4$ SYM theory (for which the β-function vanishes), the one-loop Catani operator $\mathbf{I}^{(1)}(\epsilon)$ takes the form[3]

$$\mathbf{I}^{(1)}(\epsilon) = \frac{1}{2\epsilon^2}\sum_{i=1}^{4}\sum_{j\neq i}^{4}\mathbf{T}_i\cdot\mathbf{T}_j\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon} \tag{3.6}$$

where $\mathbf{T}_i\cdot\mathbf{T}_j = T_i^a T_j^a$ and T_i^a are the SU(N) generators in the adjoint representation.

The two-loop Catani operator $\mathbf{I}^{(2)}(\epsilon)$ may be written in the case of $\mathcal{N}=4$ SYM as [28,29]

$$\mathbf{I}^{(2)}(\epsilon) = -\frac{1}{2}\left[\mathbf{I}^{(1)}(\epsilon)\right]^2 - N\zeta_2\, c(\epsilon)\ \mathbf{I}^{(1)}(2\epsilon) + \frac{c(\epsilon)}{4\epsilon}\Big[-\frac{N\zeta_3}{2}\sum_{i=1}^{4}\sum_{j\neq i}^{4}\mathbf{T}_i\cdot\mathbf{T}_j\Big(\frac{\mu^2}{-s_{ij}}\Big)^{2\epsilon} + \hat{\mathbf{H}}^{(2)}\Big] \tag{3.7}$$

where

$$c(\epsilon) \equiv \mathrm{e}^{-\epsilon\gamma}\Gamma(1-\epsilon) = \frac{\pi\epsilon}{\sin(\pi\epsilon)}\exp\left[\sum_{k=2}^{\infty}(-1)^{k+1}\frac{\zeta_k\epsilon^k}{k}\right] = 1 + \frac{\pi^2}{12}\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{3.8}$$

$$\hat{\mathbf{H}}^{(2)} = -4L\,[\mathbf{T}_1\cdot\mathbf{T}_2, \mathbf{T}_2\cdot\mathbf{T}_3] \tag{3.9}$$

$$L = \log\left(\frac{s}{t}\right)\log\left(\frac{t}{u}\right)\log\left(\frac{u}{s}\right) . \tag{3.10}$$

We may use eq. (3.6) to rewrite eq. (3.7) as

$$\mathbf{I}^{(2)}(\epsilon) = -\frac{1}{2}\left[\mathbf{I}^{(1)}(\epsilon)\right]^2 - N(\zeta_2 + \epsilon\zeta_3)\, c(\epsilon)\, \mathbf{I}^{(1)}(2\epsilon) + \frac{c(\epsilon)}{4\epsilon}\hat{\mathbf{H}}^{(2)} . \tag{3.11}$$

Using eq. (3.11), eq. (3.4) may be rewritten in the form

$$\begin{aligned}|A^{(2)}(\epsilon)\rangle &= \frac{1}{2N}\mathbf{I}^{(1)}(\epsilon)|A^{(1)}(\epsilon)\rangle - \frac{1}{N}(\zeta_2+\epsilon\zeta_3)\,c(\epsilon)\,\mathbf{I}^{(1)}(2\epsilon)|A^{(0)}\rangle \\ &\quad + \frac{1}{4N^2}\frac{c(\epsilon)}{\epsilon}\hat{\mathbf{H}}^{(2)}|A^{(0)}\rangle + \frac{1}{2N}\mathbf{I}^{(1)}(\epsilon)|A^{(1f)}(\epsilon)\rangle + |A^{(2f)}(\epsilon)\rangle\end{aligned} \tag{3.12}$$

which will be useful in sec. 4.

To determine the form of the three-loop Catani operator $\mathbf{I}^{(3)}$, we turn now to the slightly different IR analysis of Sterman and Tejeda-Yeomans [19].

[3] We adopt the normalization convention of ref. [20], omitting the prefactor $\mathrm{e}^{\epsilon\gamma}/\Gamma(1-\epsilon)$ that appears in refs. [18, 25, 28, 29]. This only affects the form of the finite contribution $|A^{(Lf)}(\epsilon)\rangle$.

3.2 Formalism of Sterman and Tejeda-Yeomans

In ref. [19], Sterman and Tejeda-Yeomans characterized the IR divergences of the L-loop amplitude $|A^{(L)}(\epsilon)\rangle$ as

$$\begin{aligned}
|A^{(1)}(\epsilon)\rangle &= \frac{1}{N}\mathbf{F}^{(1)}(\epsilon)|A^{(0)}\rangle + |\tilde{A}^{(1f)}(\epsilon)\rangle && (3.13)\\
|A^{(2)}(\epsilon)\rangle &= \frac{1}{N^2}\mathbf{F}^{(2)}(\epsilon)|A^{(0)}\rangle + \frac{1}{N}\mathbf{F}^{(1)}(\epsilon)|A^{(1)}(\epsilon)\rangle + |\tilde{A}^{(2f)}(\epsilon)\rangle && (3.14)\\
|A^{(3)}(\epsilon)\rangle &= \frac{1}{N^3}\mathbf{F}^{(3)}(\epsilon)|A^{(0)}\rangle + \frac{1}{N^2}\mathbf{F}^{(2)}(\epsilon)|A^{(1)}\rangle + \frac{1}{N}\mathbf{F}^{(1)}(\epsilon)|A^{(2)}(\epsilon)\rangle + |\tilde{A}^{(3f)}(\epsilon)\rangle && (3.15)
\end{aligned}$$

where we have rescaled[4] the operators $\mathbf{F}^{(L)}$ of ref. [19] by a factor of N^L. The operators $\mathbf{F}^{(L)}(\epsilon)$ differ from the Catani operators $\mathbf{I}^{(L)}(\epsilon)$ introduced in the previous subsection in that they contain only the divergent terms of the expansion in ϵ whereas the expansion of $\mathbf{I}^{(L)}(\epsilon)$ also contains non-negative powers of ϵ. For this reason, the finite pieces $|\tilde{A}^{(Lf)}(\epsilon)\rangle$ of eqs. (3.13)–(3.15) differ from the $|A^{(Lf)}(\epsilon)\rangle$ of eqs. (3.3)–(3.5), which is why we have distinguished them with a tilde.

Specializing to the case of $gg \to gg$ in $\mathcal{N} = 4$ SYM theory, we may rewrite the expressions for $\mathbf{F}^{(L)}(\epsilon)$ given in ref. [19] as

$$\begin{aligned}
\mathbf{F}^{(1)}(\epsilon) &= \mathbf{G}^{(1)}(\epsilon) && (3.16)\\
\mathbf{F}^{(2)}(\epsilon) &= -\frac{1}{2}\left[\mathbf{F}^{(1)}(\epsilon)\right]^2 + \mathbf{G}^{(2)}(2\epsilon) && (3.17)\\
\mathbf{F}^{(3)}(\epsilon) &= -\frac{1}{3}\left[\mathbf{F}^{(1)}(\epsilon)\right]^3 - \frac{1}{3}\mathbf{F}^{(1)}(\epsilon)\mathbf{F}^{(2)}(\epsilon) - \frac{2}{3}\mathbf{F}^{(2)}(\epsilon)\mathbf{F}^{(1)}(\epsilon) + \mathbf{G}^{(3)}(3\epsilon) && (3.18)
\end{aligned}$$

with

$$\mathbf{G}^{(L)}(\epsilon) = \frac{N^L}{2}\left[-\left(\frac{\gamma^{(L)}}{\epsilon^2} + \frac{2\mathcal{G}_0^{(L)}}{\epsilon}\right)\mathbb{1} + \frac{1}{\epsilon}\mathbf{\Gamma}^{(L)}\right] \qquad (3.19)$$

where $\mathbf{\Gamma}^{(L)}$ are nontrivial anomalous dimension matrices and $\gamma^{(L)}$ and $\mathcal{G}_0^{(L)}$ are the coefficients of the soft (or Wilson line cusp) and collinear anomalous dimensions of the gluon respectively, which are just numbers (proportional to the identity matrix)

$$\begin{aligned}
\gamma(a) &= \sum_{L=1}^{\infty}\gamma^{(L)}a^L = 4a - 4\zeta_2 a^2 + 22\zeta_4 a^3 + \cdots\\
\mathcal{G}_0(a) &= \sum_{L=1}^{\infty}\mathcal{G}_0^{(L)}a^L = -\zeta_3 a^2 + (4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3)a^3 + \cdots && (3.20)
\end{aligned}$$

These values were calculated in ref. [20] in the planar (leading N) limit, but they remain valid for arbitrary N because contributions subleading-in-$1/N$ are never proportional to the identity, and for $\mathbf{G}^{(L)}(\epsilon)$ contribute only to the anomalous dimension matrices $\mathbf{\Gamma}^{(L)}$. Moreover, $\gamma(a)$ and $\mathcal{G}_0(a)$ are the leading transcendentality part of the corresponding QCD anomalous dimensions, as one can easily check using the formulas in ref. [19].

[4]Note also that the expansion in ref. [19] is in powers of α/π whereas in eq. (3.2) the expansion is in powers of a. The only effect of this difference on the equations, however, is to change the numerical values of the constants $\gamma^{(L)}$ and $\mathcal{G}_0^{(L)}$ in eq. (3.19).

3.3 Comparison of IR formalisms

We now show that the operators defined by Catani and by Sterman/Tejeda-Yeomans are related by the following equations

$$\begin{aligned}
\mathbf{F}^{(1)}(\epsilon) &= \mathbf{I}^{(1)}(\epsilon) - \Delta^{(1)}(\epsilon) && (3.21)\\
\mathbf{F}^{(2)}(\epsilon) &= \mathbf{I}^{(2)}(\epsilon) + \Delta^{(1)}(\epsilon)\,\mathbf{I}^{(1)}(\epsilon) - \Delta^{(2)}(\epsilon) && (3.22)\\
\mathbf{F}^{(3)}(\epsilon) &= \mathbf{I}^{(3)}(\epsilon) + \Delta^{(1)}(\epsilon)\,\mathbf{I}^{(2)}(\epsilon) + \Delta^{(2)}(\epsilon)\,\mathbf{I}^{(1)}(\epsilon) - \Delta^{(3)}(\epsilon) && (3.23)
\end{aligned}$$

and

$$\begin{aligned}
|\tilde{A}^{(1f)}(\epsilon)\rangle &= |A^{(1f)}(\epsilon)\rangle + \frac{1}{N}\Delta^{(1)}(\epsilon)|A^{(0)}\rangle \\
|\tilde{A}^{(2f)}(\epsilon)\rangle &= |A^{(2f)}(\epsilon)\rangle + \frac{1}{N}\Delta^{(1)}(\epsilon)|A^{(1f)}(\epsilon)\rangle + \frac{1}{N^2}\Delta^{(2)}(\epsilon)|A^{(0)}\rangle && (3.24)\\
|\tilde{A}^{(3f)}(\epsilon)\rangle &= |A^{(3f)}(\epsilon)\rangle + \frac{1}{N}\Delta^{(1)}(\epsilon)|A^{(2f)}(\epsilon)\rangle + \frac{1}{N^2}\Delta^{(2)}(\epsilon)|A^{(1f)}(\epsilon)\rangle + \frac{1}{N^3}\Delta^{(3)}(\epsilon)|A^{(0)}\rangle
\end{aligned}$$

where the $\Delta^{(L)}(\epsilon)$ are finite as $\epsilon \to 0$.

By comparing eq. (3.6) with eq. (3.16), one may ascertain that the $\mathcal{O}(1/\epsilon^2)$ terms on both sides of eq. (3.21) match because $\sum_{i=1}^{4}\sum_{j\neq i}^{4}\mathbf{T}_i\cdot\mathbf{T}_j = -4N\mathbb{1}$. The $\mathcal{O}(1/\epsilon)$ terms match provided $\mathbf{\Gamma}^{(1)}$ is given by

$$\mathbf{\Gamma}^{(1)} = \frac{1}{N}\sum_{i=1}^{4}\sum_{j\neq i}^{4}\mathbf{T}_i\cdot\mathbf{T}_j \log\left(\frac{\mu^2}{-s_{ij}}\right). \tag{3.25}$$

The remaining (finite) part of $\mathbf{I}^{(1)}(\epsilon)$ defines $\Delta^{(1)}(\epsilon)$, the first term of which is

$$\Delta^{(1)}(\epsilon) = \frac{1}{4}\sum_{i=1}^{4}\sum_{j\neq i}^{4}\mathbf{T}_i\cdot\mathbf{T}_j \log^2\left(\frac{\mu^2}{-s_{ij}}\right) + \mathcal{O}(\epsilon)\,. \tag{3.26}$$

Next by using eq. (3.22), we see that eqs. (3.11) and (3.17) are compatible provided that[5]

$$\mathbf{\Gamma}^{(2)} = -\zeta_2\mathbf{\Gamma}^{(1)} + \frac{1}{N^2}\hat{\mathbf{H}}^{(2)} - \frac{2}{N^2}\left(\epsilon\left[\mathbf{I}^{(1)},\Delta^{(1)}\right]\right)\Big|_{\epsilon\to 0} \tag{3.27}$$

with $\Delta^{(2)}(\epsilon)$ given by the finite contribution of $\mathbf{I}^{(2)}(\epsilon) + \Delta^{(1)}(\epsilon)\,\mathbf{I}^{(1)}(\epsilon)$ in eq. (3.22).

Finally, by using eq. (3.23) together with eq. (3.18), we obtain an expression for the three-loop Catani operator, namely

$$\begin{aligned}
\mathbf{I}^{(3)}(\epsilon) &= -\frac{1}{3}\left[\mathbf{I}^{(1)}(\epsilon)\right]^3 - \frac{1}{3}\mathbf{I}^{(1)}(\epsilon)\mathbf{I}^{(2)}(\epsilon) - \frac{2}{3}\mathbf{I}^{(2)}(\epsilon)\mathbf{I}^{(1)}(\epsilon) + \mathbf{G}^{(3)}(3\epsilon) && (3.28)\\
&\quad+\frac{1}{3}\left\{\left[2\mathbf{I}^{(2)}(\epsilon) + \mathbf{I}^{(1)}(\epsilon)^2, \Delta^{(1)}(\epsilon)\right] - \left[\mathbf{I}^{(1)}(\epsilon),\Delta^{(1)}(\epsilon)\right]\Delta^{(1)}(\epsilon) + \left[\mathbf{I}^{(1)}(\epsilon),\Delta^{(2)}(\epsilon)\right]\right\}
\end{aligned}$$

[5] Further calculation shows [30,31] that $\mathbf{\Gamma}^{(2)} = -\zeta_2\mathbf{\Gamma}^{(1)}$, since the last two terms of eq. (3.27) exactly cancel, as can be seen using eqns. (D5)-(D8) of ref. [31]. We thank Lance Dixon for pointing this out.

where all the commutator terms in the second line have a leading $\mathcal{O}(1/\epsilon)$ divergence. Since we do not have an explicit expression for $\mathbf{\Gamma}^{(3)}(\epsilon)$ in $\mathbf{G}^{(3)}(\epsilon)$, we do not know the $\mathcal{O}(1/\epsilon)$ contribution anyway so we write

$$\mathbf{I}^{(3)}(\epsilon) = -\frac{1}{3}\left[\mathbf{I}^{(1)}(\epsilon)\right]^3 - \frac{1}{3}\mathbf{I}^{(1)}(\epsilon)\mathbf{I}^{(2)}(\epsilon) - \frac{2}{3}\mathbf{I}^{(2)}(\epsilon)\mathbf{I}^{(1)}(\epsilon) - \frac{11\zeta_4 N^3}{(3\epsilon)^2}\mathbb{1} + \mathcal{O}\left(\frac{1}{\epsilon}\right). \qquad (3.29)$$

In fact, we can check that the commutators in the second line of (3.28) don't have pieces proportional to the identity, thus we can absorb them inside the three-loop anomalous dimension matrix $\mathbf{\Gamma}^{(3)}$, as in eq. (3.27). Thus we can also calculate the part of the $\mathcal{O}(1/\epsilon)$ term proportional to the identity,

$$-\frac{\left(4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3\right)}{3\epsilon}N^3\mathbb{1} \qquad (3.30)$$

and we are left only with the unknown divergent piece $\mathbf{\Gamma}^{(3)}/\epsilon$.

If we substitute eq. (3.11) into eq. (3.29), we obtain

$$\begin{aligned}\mathbf{I}^{(3)}(\epsilon) \;=\; & \frac{1}{6}\left[\mathbf{I}^{(1)}(\epsilon)\right]^3 + \frac{N}{3}(\zeta_2 + \epsilon\zeta_3)c(\epsilon)\left[\mathbf{I}^{(1)}(\epsilon)\mathbf{I}^{(1)}(2\epsilon) + 2\mathbf{I}^{(1)}(2\epsilon)\mathbf{I}^{(1)}(\epsilon)\right] \\ & -\frac{c(\epsilon)}{12\epsilon}\left[\mathbf{I}^{(1)}(\epsilon)\hat{\mathbf{H}}^{(2)} + 2\hat{\mathbf{H}}^{(2)}\mathbf{I}^{(1)}(\epsilon)\right] - \frac{11\zeta_4 N^3}{9\epsilon^2}\mathbb{1} + \mathcal{O}\left(\frac{1}{\epsilon}\right). \end{aligned} \qquad (3.31)$$

Finally, we use eq. (3.29) to rewrite eq. (3.5) as

$$\begin{aligned}|A^{(3)}(\epsilon)\rangle \;=\; & \frac{2}{3N}\mathbf{I}^{(1)}(\epsilon)|A^{(2)}(\epsilon)\rangle + \frac{1}{3N^2}\mathbf{I}^{(2)}(\epsilon)|A^{(1)}(\epsilon)\rangle + \frac{1}{3N}\mathbf{I}^{(1)}(\epsilon)|A^{(2f)}(\epsilon)\rangle \\ & + \frac{2}{3N^2}\left[\mathbf{I}^{(2)}(\epsilon) + \frac{1}{2}\mathbf{I}^{(1)}(\epsilon)^2\right]|A^{(1f)}(\epsilon)\rangle - \frac{11\zeta_4}{9\epsilon^2}|A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) \end{aligned} \qquad (3.32)$$

and then use eq. (3.11) to obtain

$$\begin{aligned}|A^{(3)}(\epsilon)\rangle \;=\; & \frac{2}{3N}\mathbf{I}^{(1)}(\epsilon)|A^{(2)}(\epsilon)\rangle + \frac{1}{3N^2}\mathbf{I}^{(2)}(\epsilon)|A^{(1)}(\epsilon)\rangle \\ & -\frac{2}{3\epsilon^2}|A^{(2f)}(\epsilon)\rangle + \frac{\zeta_2}{3\epsilon^2}|A^{(1f)}(\epsilon)\rangle - \frac{11\zeta_4}{9\epsilon^2}|A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) \end{aligned} \qquad (3.33)$$

which will be useful in the following section.

4 $1/N$ expansion of the IR divergences

In this section, we will use the results of the previous section to expand the IR-divergent contributions of the four-gluon amplitude in powers of $1/N$.

First we re-express the vector of amplitudes (3.1) as

$$|A^{(L)}\rangle = \begin{pmatrix} |A^{(L,0)}\rangle + \frac{1}{N^2}|A^{(L,2)}\rangle + \cdots \\ \frac{1}{N}|A^{(L,1)}\rangle + \frac{1}{N^3}|A^{(L,3)}\rangle + \cdots \end{pmatrix} \qquad (4.1)$$

where

$$|A^{(L,2k)}\rangle = \begin{pmatrix} A^{(L,2k)}_{[1]} \\ A^{(L,2k)}_{[2]} \\ A^{(L,2k)}_{[3]} \\ A^{(L,2k)}_{[4]} \\ A^{(L,2k)}_{[5]} \\ A^{(L,2k)}_{[6]} \end{pmatrix} \qquad \text{and} \qquad |A^{(L,2k+1)}\rangle = \begin{pmatrix} A^{(L,2k+1)}_{[7]} \\ A^{(L,2k+1)}_{[8]} \\ A^{(L,2k+1)}_{[9]} \end{pmatrix} \tag{4.2}$$

We recall that the leading-color amplitude $A^{(L,0)}$ is proportional to N^L in the full amplitude $\mathcal{A}_{4-\text{gluon}}$ because it is multiplied by $a^L \sim N^L$. The subleading-color contributions $A^{(L,k)}$ are proportional to N^{L-k} in $\mathcal{A}_{4-\text{gluon}}$, with the most-subleading contribution $A^{(L,L)}$ being the N-independent piece of the amplitude.

In the basis (4.1) and (4.2), the operator $\mathbf{I}^{(1)}(\epsilon)$, defined in eq. (3.6), has the form [25]

$$\mathbf{I}^{(1)}(\epsilon) = -\frac{1}{\epsilon^2} \begin{pmatrix} N\alpha_\epsilon & \beta_\epsilon \\ \gamma_\epsilon & N\delta_\epsilon \end{pmatrix} \tag{4.3}$$

where

$$\alpha_\epsilon = \begin{pmatrix} \mathtt{S}+\mathtt{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathtt{S}+\mathtt{U} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathtt{T}+\mathtt{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathtt{T}+\mathtt{U} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathtt{S}+\mathtt{U} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathtt{S}+\mathtt{T} \end{pmatrix}, \qquad \beta_\epsilon = \begin{pmatrix} \mathtt{T}-\mathtt{U} & 0 & \mathtt{S}-\mathtt{U} \\ \mathtt{U}-\mathtt{T} & \mathtt{S}-\mathtt{T} & 0 \\ 0 & \mathtt{T}-\mathtt{S} & \mathtt{U}-\mathtt{S} \\ 0 & \mathtt{T}-\mathtt{S} & \mathtt{U}-\mathtt{S} \\ \mathtt{U}-\mathtt{T} & \mathtt{S}-\mathtt{T} & 0 \\ \mathtt{T}-\mathtt{U} & 0 & \mathtt{S}-\mathtt{U} \end{pmatrix}$$

$$\gamma_\epsilon = \begin{pmatrix} \mathtt{S}-\mathtt{U} & \mathtt{S}-\mathtt{T} & 0 & 0 & \mathtt{S}-\mathtt{T} & \mathtt{S}-\mathtt{U} \\ 0 & \mathtt{U}-\mathtt{T} & \mathtt{U}-\mathtt{S} & \mathtt{U}-\mathtt{S} & \mathtt{U}-\mathtt{T} & 0 \\ \mathtt{T}-\mathtt{U} & 0 & \mathtt{T}-\mathtt{S} & \mathtt{T}-\mathtt{S} & 0 & \mathtt{T}-\mathtt{U} \end{pmatrix}, \qquad \delta_\epsilon = \begin{pmatrix} 2\mathtt{S} & 0 & 0 \\ 0 & 2\mathtt{U} & 0 \\ 0 & 0 & 2\mathtt{T} \end{pmatrix} \tag{4.4}$$

with

$$\mathtt{S} = \left(-\frac{\mu^2}{s}\right)^\epsilon, \qquad \mathtt{T} = \left(-\frac{\mu^2}{t}\right)^\epsilon, \qquad \mathtt{U} = \left(-\frac{\mu^2}{u}\right)^\epsilon. \tag{4.5}$$

Using this together with eq. (3.9), we can compute

$$\hat{\mathbf{H}}^{(2)} = L \begin{pmatrix} h_\alpha & Nh_\beta \\ Nh_\gamma & 0 \end{pmatrix} \tag{4.6}$$

with

$$h_\alpha = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}, \qquad h_\beta = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

$$h_\gamma \;=\; \left(\begin{array}{cccccc} -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{array}\right). \tag{4.7}$$

The leading-color amplitude $|A^{(L,0)}\rangle$ has poles up to $\mathcal{O}(1/\epsilon^{2L})$, but from these expressions, we can see that each additional power of $1/N$ in the amplitude reduces the power of the leading pole in ϵ by one, so that subleading-color amplitude $|A^{(L,k)}\rangle$ only has poles up to $\mathcal{O}(1/\epsilon^{2L-k})$. The diagonal (leading in N) elements of eq. (4.3) have leading power $1/\epsilon^2$, whereas the off-diagonal (subleading in N) elements have leading power $1/\epsilon$, since β_ϵ and γ_ϵ have expansions that start at $\mathcal{O}(\epsilon)$. Since the leading divergences of $\mathbf{I}^{(2)}$ and $\mathbf{I}^{(3)}$ are given by $-\frac{1}{2}\left[\mathbf{I}^{(1)}\right]^2$ and $\frac{1}{6}\left[\mathbf{I}^{(1)}\right]^3$ respectively, one can use induction on eqs. (3.3)–(3.5) to show that the leading pole of $|A^{(L,k)}\rangle$ is $\mathcal{O}(1/\epsilon^{2L-k})$. We will see this explicitly in the following subsections.

4.1 One-loop divergences

We now substitute eqs. (4.1) and (4.3) into eq. (3.3) to obtain equations for the leading- and subleading-color one-loop amplitudes $A^{(1,0)}$ and $A^{(1,1)}$. The leading-color amplitude satisfies

$$|A^{(1,0)}(\epsilon)\rangle = -\frac{1}{\epsilon^2}\alpha_\epsilon|A^{(0)}\rangle + |A^{(1f,0)}(\epsilon)\rangle\,. \tag{4.8}$$

This equation is diagonal, so we focus on the first component

$$A^{(1,0)}_{[1]}(\epsilon) = -\frac{1}{\epsilon^2}(\mathtt{S}+\mathtt{T})A^{(0)}_{[1]}(\epsilon) + A^{(1f,0)}_{[1]}(\epsilon) \tag{4.9}$$

with the other components given by permutations of s, t, and u. The finite contribution $A^{(1f,0)}_{[1]}(\epsilon)$ is not specified by the IR analysis, but may be obtained by evaluating the exact expression (e.g., as in ref. [20]) for the amplitude (2.8)

$$\begin{aligned} A^{(1,0)}_{[1]} &= M^{(1)}(s,t)\,A^{(0)}_{[1]} \\ M^{(1)}(s,t) &= -\frac{(\mathtt{S}+\mathtt{T})}{\epsilon^2} + \frac{1}{2}\log^2\left(\frac{s}{t}\right) + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{4.10}$$

The equation for the subleading-color one-loop amplitude

$$|A^{(1,1)}(\epsilon)\rangle = -\frac{1}{\epsilon^2}\gamma_\epsilon|A^{(0)}\rangle + |A^{(1f,1)}(\epsilon)\rangle \tag{4.11}$$

is consistent with the one-loop U(1) decoupling relation (2.9), but the latter (exact) relation also allows us to evaluate the finite contribution

$$A^{(1,1)}_{[7]}(\epsilon) = A^{(1,1)}_{[8]}(\epsilon) = A^{(1,1)}_{[9]}(\epsilon) = \left(\frac{-8iK}{stu}\right)\left[\frac{(s\,\mathtt{S}+t\,\mathtt{T}+u\,\mathtt{U})}{\epsilon^2} + \frac{1}{2}\left(sX^2+tY^2+uZ^2\right) + \mathcal{O}(\epsilon)\right] \tag{4.12}$$

where we define

$$X = \log\left(\frac{t}{u}\right), \qquad Y = \log\left(\frac{u}{s}\right), \qquad Z = \log\left(\frac{s}{t}\right). \tag{4.13}$$

We now expand $\mathtt{S}$, $\mathtt{T}$, and $\mathtt{U}$ in ϵ and re-express

$$\log(-s/\mu^2) = \log(-u/\mu^2) - Y, \qquad \log(-t/\mu^2) = \log(-u/\mu^2) + X, \qquad Z = -X - Y \quad (4.14)$$

to obtain

$$|A^{(1,1)}(\epsilon)\rangle = \left(\frac{-8iK}{stu}\right)\left[\left(\frac{\mu^2}{-u}\right)^{\epsilon}\frac{(sY - tX)}{\epsilon} - (s+t)XY\right]\begin{pmatrix}1\\1\\1\end{pmatrix} + \mathcal{O}(\epsilon) \qquad (4.15)$$

where $(1,1,1)^T$ indicates the [7], [8], and [9] components of $A^{(1,1)}$. One can see that, while the leading-color one-loop amplitude (4.9) has an $\mathcal{O}(1/\epsilon^2)$ leading divergence, the subleading-color amplitude (4.15) has only an $\mathcal{O}(1/\epsilon)$ pole.

4.2 Two-loop divergences

We derive expressions for the leading- and subleading-color two-loop amplitudes $A^{(2,0)}$, $A^{(2,1)}$, and $A^{(2,2)}$ by substituting eqs. (4.1), (4.3), and (4.6) into eq. (3.12).

The IR behavior of the leading-color amplitude was utilized in ref. [1] to motivate the ABDK relation between one- and two-loop amplitudes. To see this, observe that the leading-color amplitude satisfies

$$\begin{aligned} |A^{(2,0)}(\epsilon)\rangle &= -\frac{1}{2\epsilon^2}\alpha_\epsilon\left[|A^{(1,0)}(\epsilon)\rangle + |A^{(1f,0)}(\epsilon)\rangle\right] \\ &\quad -(\zeta_2 + \epsilon\zeta_3)c(\epsilon)\left[|A^{(1,0)}(2\epsilon)\rangle - |A^{(1f,0)}(2\epsilon)\rangle\right] + |A^{(2f,0)}(\epsilon)\rangle \end{aligned} \qquad (4.16)$$

the first component of which reads

$$A^{(2,0)}_{[1]}(\epsilon) = -\frac{(\mathtt{S}+\mathtt{T})}{2\epsilon^2}\left[A^{(1,0)}_{[1]}(\epsilon) + A^{(1f,0)}_{[1]}(\epsilon)\right] - (\zeta_2+\epsilon\zeta_3)c(\epsilon)\left[A^{(1,0)}_{[1]}(2\epsilon) - A^{(1f,0)}_{[1]}(2\epsilon)\right] + A^{(2f,0)}_{[1]}(\epsilon)\,. \qquad (4.17)$$

Using eqs. (2.11) and (4.10), we may rewrite eq. (4.17) as

$$M^{(2)}(\epsilon) = \frac{1}{2}\left[M^{(1)}(\epsilon) - M^{(1f)}(\epsilon)\right]\left[M^{(1)}(\epsilon) + M^{(1f)}(\epsilon)\right] - (\zeta_2+\epsilon\zeta_3)c(\epsilon)\left[M^{(1)}(2\epsilon) - M^{(1f)}(2\epsilon)\right] + M^{(2f)}(\epsilon) \qquad (4.18)$$

where $M^{(Lf)}(\epsilon) = A^{(Lf,0)}_{[1]}(\epsilon)/A^{(0)}_{[1]}$ and we have suppressed the s, t dependence of $M^{(L)}$. Retaining only the divergent pieces, we get

$$M^{(2)}(\epsilon) = \frac{1}{2}\left[M^{(1)}(\epsilon)\right]^2 - (\zeta_2 + \epsilon\zeta_3)M^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^0)\,. \qquad (4.19)$$

Of course, the Catani equation (3.4) does not yield any information about the $\mathcal{O}(\epsilon^0)$ piece, but Anastasiou et al. showed, using the exact one- and two-loop results (2.8) and (2.12), that it is actually a constant (independent of the kinematic variables s and t), yielding [1]

$$M^{(2)}(\epsilon) = \frac{1}{2}\left[M^{(1)}(\epsilon)\right]^2 - (\zeta_2 + \epsilon\zeta_3 + \epsilon^2\zeta_4)M^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon)\,. \qquad (4.20)$$

The leading divergence of $M^{(2)}(\epsilon)$ is $\mathcal{O}(1/\epsilon^4)$ as expected.

Next, the equation for the two-loop double-trace amplitude is

$$
\begin{aligned}
|A^{(2,1)}(\epsilon)\rangle &= -\frac{1}{2\epsilon^2}\gamma_\epsilon\Big[|A^{(1,0)}(\epsilon)\rangle + |A^{(1f,0)}(\epsilon)\rangle\Big] - \frac{1}{2\epsilon^2}\delta_\epsilon\Big[|A^{(1,1)}(\epsilon)\rangle + |A^{(1f,1)}(\epsilon)\rangle\Big] \\
&\quad -(\zeta_2+\epsilon\zeta_3)c(\epsilon)\Big[|A^{(1,1)}(2\epsilon)\rangle - |A^{(1f,1)}(2\epsilon)\rangle\Big] + \frac{c(\epsilon)}{4\epsilon}Lh_\gamma|A^{(0)}\rangle + |A^{(2f,1)}(\epsilon)\rangle \qquad (4.21)\\
&= \left(\frac{-8iK}{stu}\right)\frac{(-2)(sY-tX)}{\epsilon^3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \qquad (4.22)
\end{aligned}
$$

whose leading pole is $\mathcal{O}(1/\epsilon^3)$.

Finally, the N-independent single-trace amplitude satisfies

$$
|A^{(2,2)}(\epsilon)\rangle = -\frac{1}{2\epsilon^2}\beta_\epsilon\Big[|A^{(1,1)}(\epsilon)\rangle + |A^{(1f,1)}(\epsilon)\rangle\Big] + \frac{c(\epsilon)}{4\epsilon}Lh_\alpha|A^{(0)}\rangle + |A^{(2f,2)}(\epsilon)\rangle\,. \qquad (4.23)
$$

Using eqs. (4.4) and (4.6) and the first component of eq. (4.23) one obtains

$$
A^{(2,2)}_{[1]}(\epsilon) = -\frac{(\mathtt{S}+\mathtt{T}-2\mathtt{U})}{2\epsilon^2}\Big[A^{(1,1)}_{[7]}(\epsilon)+A^{(1f,1)}_{[7]}(\epsilon)\Big]+\frac{c(\epsilon)}{2\epsilon}L\Big[A^{(0)}_{[2]}(\epsilon)-A^{(0)}_{[3]}(\epsilon)\Big]+A^{(2f,2)}_{[1]}(\epsilon)\,. \qquad (4.24)
$$

Next we use eqs. (2.6) and (4.12) and expand in ϵ to obtain the unexpectedly simple result (due to cancellations between the $\Big[\mathbf{I}^{(1)}(\epsilon)\Big]^2$ and $\hat{\mathbf{H}}^{(2)}$ terms in eq. (3.11))

$$
A^{(2,2)}_{[1]}(\epsilon) = \left(\frac{-8iK}{stu}\right)\frac{X-Y}{\epsilon}\left[\left(\frac{\mu^2}{-u}\right)^{2\epsilon}\frac{sY-tX}{2\epsilon} - (s+t)XY\right] + \mathcal{O}(\epsilon^0) \qquad (4.25)
$$

which has an $\mathcal{O}(1/\epsilon^2)$ leading divergence. Comparing this with eq. (4.15), one obtains the following relation between the N-independent one- and two-loop amplitudes

$$
|A^{(2,2)}(\epsilon)\rangle = \frac{1}{\epsilon}\begin{pmatrix}X-Y\\Z-X\\Y-Z\\Y-Z\\Z-X\\X-Y\end{pmatrix}A^{(1,1)}_{[7]}(2\epsilon) + \mathcal{O}(\epsilon^0)\,. \qquad (4.26)
$$

This relation manifestly obeys eq. (2.17). We have also verified eq. (4.26) using the exact two-loop amplitude (2.14). Unlike the case of the ABDK relation (4.20) between leading-color one- and two-loop amplitudes, however, the $\mathcal{O}(\epsilon^0)$ term in eq. (4.26) is not a simple constant, but rather a complicated linear combination of polylogarithms.

Finally, one can check that the equations (4.16), (4.21), and (4.23) are consistent with the group theory relations (2.16).

4.3 Three-loop divergences

We derive expressions for the leading- and subleading-color three-loop amplitudes $A^{(3,0)}$, $A^{(3,1)}$, $A^{(3,2)}$, and $A^{(3,3)}$ by substituting $\mathbf{I}^{(2)}$ from (3.11), $\mathbf{I}^{(1)}$ from (4.3), and $\hat{\mathbf{H}}^{(2)}$ from (4.6) into eq. (3.33).

The leading-color three-loop amplitude obeys

$$\begin{aligned} |A^{(3,0)}(\epsilon)\rangle &= -\frac{2}{3\epsilon^2}\alpha_\epsilon|A^{(2,0)}(\epsilon)\rangle - \frac{1}{6\epsilon^4}\alpha_\epsilon^2|A^{(1,0)}(\epsilon)\rangle + \frac{(\zeta_2+\epsilon\zeta_3)c(\epsilon)}{12\epsilon^2}\alpha_{2\epsilon}|A^{(1,0)}(\epsilon)\rangle \\ &\quad -\frac{2}{3\epsilon^2}|A^{(2f,0)}(\epsilon)\rangle + \frac{\zeta_2}{3\epsilon^2}|A^{(1f,0)}(\epsilon)\rangle - \frac{11\zeta_4}{9\epsilon^4}|A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right). \end{aligned} \tag{4.27}$$

This equation may be shown to imply that

$$M^{(3)}(\epsilon) = M^{(1)}(\epsilon)M^{(2)}(\epsilon) - \frac{1}{3}\left[M^{(1)}(\epsilon)\right]^3 - \left(\frac{11\zeta_4}{9\epsilon^2}\right) + \mathcal{O}\left(\frac{1}{\epsilon}\right) \tag{4.28}$$

which is consistent with eq. (4.4) of ref. [20], though of course not as strong, since eq. (4.28) was derived from the IR behavior whereas the result of ref. [20] was derived by evaluating the exact three-loop amplitude.

The double-trace amplitude proportional to N^2 satisfies

$$\begin{aligned} |A^{(3,1)}(\epsilon)\rangle &= -\frac{2}{3\epsilon^2}\delta_\epsilon|A^{(2,1)}(\epsilon)\rangle - \frac{2}{3\epsilon^2}\gamma_\epsilon|A^{(2,0)}(\epsilon)\rangle - \frac{1}{6\epsilon^4}\delta_\epsilon^2|A^{(1,1)}(\epsilon)\rangle \\ &\quad -\frac{1}{6\epsilon^4}\left(\gamma_\epsilon\alpha_\epsilon + \delta_\epsilon\gamma_\epsilon\right)|A^{(1,0)}(\epsilon)\rangle + \frac{(\zeta_2+\epsilon\zeta_3)c(\epsilon)}{12\epsilon^2}\delta_{2\epsilon}|A^{(1,1)}(\epsilon)\rangle \\ &\quad +\frac{(\zeta_2+\epsilon\zeta_3)c(\epsilon)}{12\epsilon^2}\gamma_{2\epsilon}|A^{(1,0)}(\epsilon)\rangle + \frac{c(\epsilon)L}{12\epsilon}h_\gamma|A^{(1,0)}(\epsilon)\rangle - \frac{2}{3\epsilon^2}|A^{(2f,1)}(\epsilon)\rangle \\ &\quad +\frac{\zeta_2}{3\epsilon^2}|A^{(1f,1)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) \\ &= \left(\frac{-8iK}{stu}\right)\frac{2(sY-tX)}{\epsilon^5}\begin{pmatrix}1\\1\\1\end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^4}\right). \end{aligned} \tag{4.29}$$

The subleading-color single-trace amplitude satisfies

$$\begin{aligned} |A^{(3,2)}(\epsilon)\rangle &= -\frac{2}{3\epsilon^2}\alpha_\epsilon|A^{(2,2)}(\epsilon)\rangle - \frac{2}{3\epsilon^2}\beta_\epsilon|A^{(2,1)}(\epsilon)\rangle - \frac{1}{6\epsilon^4}\left(\alpha_\epsilon\beta_\epsilon + \beta_\epsilon\delta_\epsilon\right)|A^{(1,1)}(\epsilon)\rangle \\ &\quad -\frac{1}{6\epsilon^4}\beta_\epsilon\gamma_\epsilon|A^{(1,0)}(\epsilon)\rangle + \frac{(\zeta_2+\epsilon\zeta_3)c(\epsilon)}{12\epsilon^2}\beta_{2\epsilon}|A^{(1,1)}(\epsilon)\rangle + \frac{c(\epsilon)L}{12\epsilon}h_\beta|A^{(1,1)}(\epsilon)\rangle \\ &\quad +\frac{c(\epsilon)L}{12\epsilon}h_\alpha|A^{(1,0)}(\epsilon)\rangle - \frac{2}{3\epsilon^2}|A^{(2f,2)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) \\ &= \left(\frac{-8iK}{stu}\right)\frac{(-1)(sY-tX)}{\epsilon^4}\begin{pmatrix}X-Y\\Z-X\\Y-Z\\Y-Z\\Z-X\\X-Y\end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right). \end{aligned} \tag{4.30}$$

Finally, the N-independent double-trace amplitude obeys

$$\begin{aligned} |A^{(3,3)}(\epsilon)\rangle &= -\frac{2}{3\epsilon^2}\gamma_\epsilon|A^{(2,2)}(\epsilon)\rangle - \frac{1}{6\epsilon^4}\gamma_\epsilon\beta_\epsilon|A^{(1,1)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) \\ &= \frac{1}{6\epsilon^4}\gamma_\epsilon\beta_\epsilon\left[|A^{(1,1)}(\epsilon)\rangle + 2|A^{(1f,1)}(\epsilon)\rangle\right] - \frac{c(\epsilon)L}{6\epsilon^3}\gamma_\epsilon h_\alpha|A^{(0)}\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) . \end{aligned} \tag{4.31}$$

We use eq. (4.15) together with $L = -XY(X+Y)$ and

$$\gamma_\epsilon\beta_\epsilon \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2\left[(S-T)^2 + (T-U)^2 + (U-S)^2\right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.32}$$

and expand in ϵ to obtain the leading two terms in the Laurent expansion

$$|A^{(3,3)}(\epsilon)\rangle = \left(\frac{-8iK}{stu}\right) \frac{X^2+Y^2+Z^2}{\epsilon^2} \left[\left(\frac{\mu^2}{-u}\right)^{3\epsilon} \frac{sY-tX}{3\epsilon} - (s+t)XY\right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon}\right) . \tag{4.33}$$

This can be written concisely as

$$|A^{(3,3)}(\epsilon)\rangle = \frac{X^2+Y^2+Z^2}{\epsilon^2}|A^{(1,1)}(3\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon}\right) . \tag{4.34}$$

One can see from all these expressions that $|A^{(3,k)}\rangle$ has a leading pole of $\mathcal{O}(1/\epsilon^{6-k})$, as expected.

4.4 Higher-loop divergences

Equations (3.11) and (3.31) suggest that the most-divergent contribution of the L-loop amplitude is given by

$$|A^{(L)}(\epsilon)\rangle = \frac{1}{L!}\left[\frac{\mathbf{I}^{(1)}(\epsilon)}{N}\right]^L |A^{(0)}\rangle + \cdots \tag{4.35}$$

which of course can be summed to give

$$|A(\epsilon)\rangle = \exp\left[\frac{\mathbf{I}^{(1)}(\epsilon)}{N}\right] |A^{(0)}\rangle + \cdots \tag{4.36}$$

Equation (4.35) is certainly valid for the leading-color contribution $|A^{(L,0)}\rangle$, as it implies

$$M^{(L)} = \frac{1}{L!}\left[M^{(1)}\right]^L + \cdots \tag{4.37}$$

the leading-term of the BDS relation [20]. Our calculations in previous subsections, however, show that eq. (4.35) also correctly gives the most-divergent $\mathcal{O}(1/\epsilon^{2L-k})$ contribution of the subleading-amplitudes $|A^{(L,k)}\rangle$, at least for $L \leq 3$. We expect this pattern to continue to

higher loops. For example, the leading divergence of the N-independent amplitude $A^{(L,L)}$ should be given by

$$\frac{1}{N^L L!}\left[\mathbf{I}^{(1)}(\epsilon)\Big|_{\text{N-indep}}\right]^L = \frac{(-1)^L}{N^L\, L!\, \epsilon^{2L}}\begin{pmatrix} 0 & \beta_\epsilon \\ \gamma_\epsilon & 0 \end{pmatrix}^L + \mathcal{O}\left(\frac{1}{\epsilon^{L-1}}\right) \tag{4.38}$$

where γ_ϵ and β_ϵ are of $\mathcal{O}(\epsilon)$, so that the leading divergence is of $\mathcal{O}(1/\epsilon^L)$.

We now treat the cases $L = 2k+1$ and $L = 2k+2$ separately. For $L = 2k+1$, using eqs. (4.15) and (4.32), eq. (4.35) implies that the leading divergence of $A^{(2k+1,2k+1)}(\epsilon)$ is given by

$$\begin{aligned} |A^{(2k+1,2k+1)}(\epsilon)\rangle &= \frac{1}{(2k+1)!\,\epsilon^{4k}}(\gamma_\epsilon\beta_\epsilon)^k|A^{(1,1)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2k}}\right) \\ &= \frac{2^k}{(2k+1)!}\left(\frac{-8iK}{stu}\right)\left[\frac{X^2+Y^2+Z^2}{\epsilon^2}\right]^k\left[\frac{sY-tX}{\epsilon}\right]\begin{pmatrix}1\\1\\1\end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^{2k}}\right) \end{aligned} \tag{4.39}$$

which can be formally summed to give

$$\sum_{k=0}^{\infty}\left(\frac{a}{N}\right)^{2k+1}|A^{(2k+1,2k+1)}(\epsilon)\rangle = \left(\frac{-8iK}{stu}\right)\sinh\left(\frac{a\sqrt{2(X^2+Y^2+Z^2)}}{N\epsilon}\right)\left[\frac{sY-tX}{\sqrt{2(X^2+Y^2+Z^2)}}\right]\begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{4.40}$$

For $2k+2$, there is one more $\mathbf{I}^{(1)}$ matrix acting, and we get

$$|A^{(2k+2,2k+2)}(\epsilon)\rangle = -\frac{1}{(2k+2)!\,\epsilon^{4k+2}}\beta_\epsilon(\gamma_\epsilon\beta_\epsilon)^k|A^{(1,1)}(\epsilon)\rangle + \mathcal{O}\left(\frac{1}{\epsilon^{2k+1}}\right) \tag{4.41}$$

and using

$$\beta_\epsilon\begin{pmatrix}1\\1\\1\end{pmatrix} = \epsilon\begin{pmatrix}Y-X\\X-Z\\Z-Y\\Z-Y\\X-Z\\Y-X\end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{4.42}$$

we find that the leading divergence of $A^{(2k+2,2k+2)}(\epsilon)$ is given by

$$|A^{(2k+2,2k+2)}(\epsilon)\rangle = \frac{2^k}{(2k+2)!}\left(\frac{-8iK}{stu}\right)\frac{1}{\epsilon}\left[\frac{X^2+Y^2+Z^2}{\epsilon^2}\right]^k\left[\frac{sY-tX}{\epsilon}\right]\begin{pmatrix}X-Y\\Z-X\\Y-Z\\Y-Z\\Z-X\\X-Y\end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^{2k+1}}\right) \tag{4.43}$$

which can also be formally summed to give

$$\sum_{k=0}^{\infty}\left(\frac{a}{N}\right)^{2k+2}|A^{(2k+2,2k+2)}(\epsilon)\rangle = \left(\frac{-8iK}{stu}\right)\left[\frac{sY-tX}{2(X^2+Y^2+Z^2)}\right]\times \tag{4.44}$$

$$\left[\cosh\left(\frac{a\sqrt{2(X^2+Y^2+Z^2)}}{N\epsilon}\right)-1\right]\begin{pmatrix} X-Y \\ Z-X \\ Y-Z \\ Y-Z \\ Z-X \\ X-Y \end{pmatrix}.$$

Based on the forms of eqs. (4.26) and (4.34), we make the stronger conjecture that the first two terms in the Laurent expansions of the N-independent amplitudes are given by

$$|A^{(2k+1,2k+1)}(\epsilon)\rangle = \frac{2^k}{(2k)!}\left[\frac{X^2+Y^2+Z^2}{\epsilon^2}\right]^k A^{(1,1)}_{[7]}((2k+1)\epsilon)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^{2k-1}}\right) \tag{4.45}$$

$$|A^{(2k+2,2k+2)}(\epsilon)\rangle = \frac{2^k}{(2k+1)!}\frac{1}{\epsilon}\left[\frac{X^2+Y^2+Z^2}{\epsilon^2}\right]^k A^{(1,1)}_{[7]}((2k+2)\epsilon)\begin{pmatrix} X-Y \\ Z-X \\ Y-Z \\ Y-Z \\ Z-X \\ X-Y \end{pmatrix} + \mathcal{O}\left(\frac{1}{\epsilon^{2k}}\right) \tag{4.46}$$

but we have not tried to verify these. These equations of course would imply that

$$\sum_{i=7}^{9} A^{(2k+1,2k+1)}_{[i]}(\epsilon) = 3\frac{2^k}{(2k)!}\left[\frac{X^2+Y^2+Z^2}{\epsilon^2}\right]^k A^{(1,1)}_{[7]}((2k+1)\epsilon) + \mathcal{O}\left(\frac{1}{\epsilon^{2k-1}}\right) \tag{4.47}$$

$$\sum_{i=1}^{6} A^{(2k+2,2k+2)}_{[i]}(\epsilon) = \mathcal{O}\left(\frac{1}{\epsilon^{2k}}\right). \tag{4.48}$$

The exponentiation property (4.36), which implies that the leading L-loop divergence of the N-independent amplitudes is $\mathcal{O}(1/\epsilon^L)$, reminds us of similar behavior in $\mathcal{N}=8$ supergravity [22, 23], so it is natural to try to relate the N-independent $\mathcal{N}=4$ SYM amplitudes to $\mathcal{N}=8$ supergravity amplitudes.

5 $\mathcal{N}=4$ SYM / $\mathcal{N}=8$ supergravity connection

In this section, we demonstrate the existence of some relations between $\mathcal{N}=4$ SYM amplitudes and $\mathcal{N}=8$ supergravity amplitudes at the one- and two-loop levels. The L-loop N-independent SYM amplitude $A^{(L,L)}$ is related to the L-loop supergravity amplitude, as both have $\mathcal{O}(1/\epsilon^L)$ leading IR divergences. Other subleading-color SYM amplitudes $A^{(L,k)}$ have $\mathcal{O}(1/\epsilon^{2L-k})$ leading IR divergences, and consequently satisfy relations involving lower-loop supergravity amplitudes.

In this section we use the notation[6]

$$A^{(L,2k)}_{\rm SYM}(s,t) = a^L A^{(L,2k)}_{[1]}, \qquad A^{(L,2k+1)}_{\rm SYM}(s,t) = -\frac{a^L}{\sqrt{2}}A^{(L,2k+1)}_{[8]} \tag{5.1}$$

[6]The normalization of $A^{(L,2k+1)}_{\rm SYM}(s,t)$ is arbitrary. We have chosen one that is most natural in the context of the SYM/supergravity relations presented in this section.

noting that the other components $A^{(L,k)}_{[i]}$ are obtained by permutations of s, t, and u. However, we omit the argument (s,t) for functions that are completely symmetric under permutations of s, t, and u. We also define

$$M^{(L,k)}_{\rm SYM}(s,t) = \frac{A^{(L,k)}_{\rm SYM}(s,t)}{A^{(0)}_{\rm SYM}(s,t)}\,. \tag{5.2}$$

Note that the coupling constant a^L is now included in the definition of $M^{(L,k)}_{\rm SYM}(s,t)$ (as it is for the supergravity amplitudes in ref. [22]) in order to make the supergravity–nonplanar SYM relations more transparent. This differs from $M^{(L)}$ defined in previous sections, so that $M^{(L,0)}_{\rm SYM} = a^L M^{(L)}$.

5.1 One- and two-loop relations

Recall that the one-loop N-independent SYM four-gluon amplitude is given by (2.10)

$$A^{(1,1)}_{\rm SYM} = -\frac{a}{\sqrt{2}} A^{(1,1)}_{[8]} = -2\sqrt{2} i K \left[\frac{g^2 N}{8\pi^2}\left(4\pi {\rm e}^{-\gamma}\right)^{\epsilon}\right]\left[I^{(1)}_4(s,t) + I^{(1)}_4(t,u) + I^{(1)}_4(u,s)\right]. \tag{5.3}$$

The one-loop supergravity four-graviton amplitude[7] may be expressed as [6, 27]

$$A^{(1)}_{\rm SG} = 8 i K^2 \left[\frac{(\kappa/2)^2}{8\pi^2}\left(4\pi {\rm e}^{-\gamma}\right)^{\epsilon}\right]\left[I^{(1)}_4(s,t) + I^{(1)}_4(t,u) + I^{(1)}_4(u,s)\right]. \tag{5.4}$$

The supergravity amplitude is proportional to K^2 rather than K due to the KLT relations [32] (a manifestation of the relation "closed string = (open string)^2"). Defining $\lambda_{\rm SYM} = g^2 N$ and $\lambda_{\rm SG} = (\kappa/2)^2$, one observes that the one-loop SYM and supergravity amplitudes are related by

$$A^{(1,1)}_{\rm SYM} = -\frac{1}{2\sqrt{2}K}\frac{\lambda_{\rm SYM}}{\lambda_{\rm SG}} A^{(1)}_{\rm SG}\,. \tag{5.5}$$

By factoring out the tree amplitudes in both the supergravity and SYM amplitudes

$$A^{(1)}_{\rm SG} = A^{(0)}_{\rm SG} M^{(1)}_{\rm SG} = \left(\frac{16 i K^2}{stu}\right) M^{(1)}_{\rm SG} \tag{5.6}$$

$$A^{(1,1)}_{\rm SYM} = A^{(0)}_{\rm SYM}(s,t) M^{(1,1)}_{\rm SYM}(s,t) = \left(-\frac{4iK}{st}\right) M^{(1,1)}_{\rm SYM}(s,t) \tag{5.7}$$

we can express eq. (5.5) in the form

$$M^{(1,1)}_{\rm SYM}(s,t) = \sqrt{2}\,\frac{\lambda_{\rm SYM}}{\lambda_{\rm SG} u} M^{(1)}_{\rm SG}\,. \tag{5.8}$$

In other words, the ratio of the one-loop subleading-color SYM and the one-loop supergravity amplitudes (after factoring out the tree amplitudes) is simply proportional to the ratio

[7] after stripping off a factor of $(\kappa/2)^2$ for a four-point amplitude

of coupling constants, where we need to use the effective dimensionless coupling $\lambda_{\rm SG} u$ for supergravity because $\lambda_{\rm SG}$ is dimensionful.

Finally, we rewrite eq. (5.8) in the manifestly permutation-symmetric form

$$\frac{1}{3}\Big[(\lambda_{\rm SG} u) M_{\rm SYM}^{(1,1)}(s,t) + \text{c.p.}\Big] = \sqrt{2}\lambda_{\rm SYM} M_{\rm SG}^{(1)} \tag{5.9}$$

(where c.p. denotes cyclic permutations of s, t, and u) even though $u M_{\rm SYM}^{(1,1)}(s,t)$ is already symmetric under permutations. A similar symmetrized relation can be written for the one-loop leading-color amplitude

$$(\lambda_{\rm SG} u) M_{\rm SYM}^{(1,0)}(s,t) + \text{c.p.} = -\lambda_{\rm SYM} M_{\rm SG}^{(1)} \tag{5.10}$$

obtained from the one-loop decoupling relation (2.9) together with eq. (5.8).

We now turn to *two loops*. First, we exhibit some relations between SYM and supergravity amplitudes that hold only for the IR-divergent terms. The easiest case to analyze is the two-loop N-independent SYM amplitude $A_{\rm SYM}^{(2,2)}(s,t)$, since from eq. (4.26) we have

$$A_{\rm SYM}^{(2,2)}(s,t) = a^2 A_{[1]}^{(2,2)} = -\sqrt{2} a \frac{X-Y}{\epsilon} A_{\rm SYM}^{(1,1)}(2\epsilon) + \mathcal{O}(\epsilon^0)\,. \tag{5.11}$$

Using eq. (5.5), we can rewrite this as

$$A_{\rm SYM}^{(2,2)}(s,t) = \frac{a}{2K}\frac{\lambda_{\rm SYM}}{\lambda_{\rm SG}}\left(\frac{X-Y}{\epsilon}\right) A_{\rm SG}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^0) \tag{5.12}$$

or equivalently

$$M_{\rm SYM}^{(2,2)}(s,t) = -2a\,\frac{\lambda_{\rm SYM}}{\lambda_{\rm SG} u}\left(\frac{X-Y}{\epsilon}\right) M_{\rm SG}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon^0) \tag{5.13}$$

thus obtaining a relation to the one-loop supergravity amplitude.

The IR-divergent part of $A_{\rm SYM}^{(2,1)}(s,t)$ is also related to the one-loop supergravity amplitude, though in a rather more complicated way. First, it can be related to the one-loop subleading-color amplitude $A_{\rm SYM}^{(1,1)}(\epsilon)$ by the expression

$$A_{\rm SYM}^{(2,1)}(s,t) = a\left(\frac{\mu^2}{t}\right)^{\epsilon}\left[\left(-\frac{2}{\epsilon^2}+\frac{7\pi^2}{12}\right)A_{\rm SYM}^{(1,1)}(\epsilon) - \frac{1}{2\epsilon^2}\left(A_{\rm SYM}^{(1,1)}\Big|_{\mathcal{O}(\epsilon^0)}\right)\right] - \frac{3aX}{\epsilon}A_{\rm SYM}^{(1,1)}(2\epsilon) + \mathcal{O}(\epsilon^0) \tag{5.14}$$

obtained from an explicit evaluation of the IR-divergent part of the scalar integrals in eq. (2.13) in the physical region $t>0$, $s,u<0$. Then, by virtue of eq. (5.5), it can be related to the one-loop supergravity amplitude by

$$\begin{aligned} M_{\rm SYM}^{(2,1)}(s,t) \;=\; \sqrt{2}\frac{a\lambda_{\rm SYM}}{\lambda_{\rm SG} u}\bigg\{ &\left(\frac{\mu^2}{t}\right)^{\epsilon}\left[\left(-\frac{2}{\epsilon^2}+\frac{7\pi^2}{12}\right)M_{\rm SG}^{(1)}(\epsilon) - \frac{1}{2\epsilon^2}\left(M_{\rm SG}^{(1)}\Big|_{\mathcal{O}(\epsilon^0)}\right)\right] \\ &- \frac{3X}{\epsilon}M_{\rm SG}^{(1)}(2\epsilon)\bigg\} + \mathcal{O}(\epsilon^0)\,. \end{aligned} \tag{5.15}$$

Next, we exhibit some two-loop relations to $\mathcal{N}=8$ supergravity that include the finite terms. First, we consider the two-loop N-independent amplitude $M_{\rm SYM}^{(2,2)}$. Multiplying eq. (5.13) by u^2 and summing over cyclic permutations, we can write

$$\frac{1}{3}\Big[(\lambda_{\rm SG} u)^2 M_{\rm SYM}^{(2,2)}(s,t) + \text{c.p.}\Big] = \lambda_{\rm SYM}^2\left[M_{\rm SG}^{(1)}(\epsilon)\right]^2 + \mathcal{O}(\epsilon^0)\,. \tag{5.16}$$

Then, using the relation $M_{\rm SG}^{(2)}(\epsilon) = \frac{1}{2}[M_{\rm SG}^{(1)}(\epsilon)]^2 + \mathcal{O}(\epsilon^0)$ between the one- and two-loop supergravity amplitudes [22, 23], we can write this as

$$\frac{1}{3}\left[(\lambda_{\rm SG}u)^2 M_{\rm SYM}^{(2,2)}(s,t) + {\rm c.p.}\right] = 2\lambda_{\rm SYM}^2 M_{\rm SG}^{(2)} \tag{5.17}$$

where we omit the $\mathcal{O}(\epsilon^0)$ because, in fact, this relation is exact (!), as may be easily verified by using the exact expression (2.14) for the N-independent SYM amplitude [5]

$$\begin{aligned} M_{\rm SYM}^{(2,2)}(s,t) &= \frac{a^2 st}{2}\Big[s\left(I_4^{(2)P}(s,t) + I_4^{(2)NP}(s,t) + I_4^{(2)P}(s,u) + I_4^{(2)NP}(s,u)\right) \\ &\quad +t\left(I_4^{(2)P}(t,s) + I_4^{(2)NP}(t,s) + I_4^{(2)P}(t,u) + I_4^{(2)NP}(t,u)\right) \\ &\quad -2u\left(I_4^{(2)P}(u,s) + I_4^{(2)NP}(u,s) + I_4^{(2)P}(u,t) + I_4^{(2)NP}(u,t)\right)\Big] \end{aligned} \tag{5.18}$$

and that for the two-loop supergravity amplitude [6]

$$M_{\rm SG}^{(2)} = -\frac{s^3 tu}{4}\left[\frac{(\kappa/2)^2}{8\pi^2}\left(4\pi {\rm e}^{-\gamma}\right)^{\epsilon}\right]^2 [I_4^{(2)P}(s,t) + I_4^{(2)NP}(s,t) + I_4^{(2)P}(s,u) + I_4^{(2)NP}(s,u)] + {\rm c.p.} \tag{5.19}$$

Finally we turn to the two-loop subleading-color amplitude $M_{\rm SYM}^{(2,1)}$. The two-loop decoupling relation (2.18) can be rewritten as

$$-\sqrt{2}\left[uM_{\rm SYM}^{(2,1)}(s,t) + {\rm c.p.}\right] = 6\left[uM_{\rm SYM}^{(2,0)}(s,t) + {\rm c.p.}\right]. \tag{5.20}$$

Using the ABDK relation [1]

$$M_{\rm SYM}^{(2,0)}(\epsilon) = \frac{1}{2}\left[M_{\rm SYM}^{(1,0)}(\epsilon)\right]^2 + af^{(2)}(\epsilon)M_{\rm SYM}^{(1,0)}(2\epsilon) + \mathcal{O}(\epsilon), \qquad f^{(2)}(\epsilon) = -(\zeta_2 + \epsilon\zeta_3 + \epsilon^2\zeta_4) \tag{5.21}$$

together with eq. (5.10), we can rewrite eq. (5.20) as

$$\begin{aligned} \frac{1}{3}\left[(\lambda_{\rm SG}u)M_{\rm SYM}^{(2,1)}(s,t) + {\rm c.p.}\right] &= -\frac{1}{\sqrt{2}}\left\{(\lambda_{\rm SG}u)\left[M_{\rm SYM}^{(1,0)}(s,t)\right]^2 + {\rm c.p.}\right\} \\ &\quad +\sqrt{2}\frac{\lambda_{\rm SYM}^2}{8\pi^2}\left(4\pi{\rm e}^{-\gamma}\right)^{\epsilon} f^{(2)}(\epsilon)M_{\rm SG}^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{5.22}$$

Unlike the previous relation, however, eq. (5.22) only holds through $\mathcal{O}(\epsilon^0)$. As mentioned above, because of the fact that the leading IR divergence of $M_{\rm SYM}^{(2,1)}$ is $\mathcal{O}(1/\epsilon^3)$, we found a relation to the one-loop supergravity amplitude rather than the two-loop one.

In the next subsection, we attempt to generalize the exact relations (5.9) and (5.17) to L loops. We might try to generalize eq. (5.22) as well, but it was based on the two-loop decoupling relation (2.18), which does not hold beyond two loops.

5.2 Looking for a general ansatz and a connection to the 't Hooft picture

The relations (5.9) and (5.17) between N-independent SYM amplitudes and supergravity amplitudes at one and two loops suggest the appealing generalization

$$\frac{1}{3}\left[(\lambda_{\rm SG}u)^L M_{\rm SYM}^{(L,L)}(s,t) + {\rm c.p.}\right] \quad \stackrel{?}{=} \quad (\sqrt{2}\lambda_{\rm SYM})^L M_{\rm SG}^{(L)} \tag{5.23}$$

which is exact for $L = 0$, 1, and 2. It is tempting to hope that this relation holds for all L. Unfortunately, eq. (5.23) fails starting at $L = 3$, even at leading order, $\mathcal{O}(1/\epsilon^L)$.

In sec. 4.3, we used the three-loop formula of Sterman and Tejeda-Yeomans to derive the first two terms in the Laurent expansion of $A^{(3,3)}$ in eqs. (4.33) and (4.34). On the supergravity side, we expect [22] that, at least at leading IR order, we have an exponentiation formula, i.e.

$$M_{\rm SG}^{(L)} = \frac{1}{L!}\left[M_{\rm SG}^{(1)}\right]^L + \mathcal{O}\left(\frac{1}{\epsilon^{L-1}}\right) = \frac{1}{L!}\left[\frac{-\lambda_{\rm SG}(sY - tX)}{8\pi^2\epsilon}\right]^L + \mathcal{O}\left(\frac{1}{\epsilon^{L-1}}\right). \tag{5.24}$$

One can explicitly check that, if eq. (5.24) is true for $L = 3$, then eq. (5.23) is not satisfied at three loops.[8]

Assuming that eq. (5.24) for the leading IR divergence of the supergravity amplitude is correct, and that the leading IR divergences of $A^{(L,L)}$ conjectured in eqs. (4.39) and (4.43) are also correct, one can show that the following relations hold:

$$\begin{aligned}
&\left[\lambda_{\rm SG}^2\frac{s^2+t^2+u^2}{3}\right]^k\frac{1}{3}\left[(\lambda_{\rm SG}u)M_{\rm SYM}^{(2k+1,2k+1)}(s,t;\epsilon) + \text{c.p.}\right]\\
&= \lambda_{\rm SYM}^{2k+1}\frac{2^{2k+1/2}}{(2k+1)!}\left[M_{\rm SG}^{(2)}(\epsilon) + \frac{1}{6}\left(\frac{\lambda_{\rm SG}}{8\pi^2}\right)^2\left(\frac{sX+tY+uZ}{\epsilon}\right)^2\right]^k M_{\rm SG}^{(1)}(\epsilon) + \mathcal{O}\left(\frac{1}{\epsilon^{2k}}\right)
\end{aligned} \tag{5.25}$$

for $L = 2k + 1$ and

$$\begin{aligned}
&\left[\lambda_{\rm SG}^2\frac{s^2+t^2+u^2}{3}\right]^k\frac{1}{3}\left[(\lambda_{\rm SG}u)^2M_{\rm SYM}^{(2k+2,2k+2)}(s,t;\epsilon) + \text{c.p.}\right]\\
&= \lambda_{\rm SYM}^{2k+2}\frac{2^{2k+2}}{(2k+2)!}\left[M_{\rm SG}^{(2)}(\epsilon) + \frac{1}{6}\left(\frac{\lambda_{\rm SG}}{8\pi^2}\right)^2\left(\frac{sX+tY+uZ}{\epsilon}\right)^2\right]^k M_{\rm SG}^{(2)}(\epsilon) + \mathcal{O}\left(\frac{1}{\epsilon^{2k+1}}\right)
\end{aligned} \tag{5.26}$$

for $L = 2k + 2$ (where $k = 0, 1, 2, \ldots$) instead of the result eq. (5.23) without the correction to $M_{\rm SG}^{(2)}$ inside the square brackets, and with $[\frac{1}{3}(s^2 + t^2 + u^2)]^k$ replaced by u^{2k}.

An interesting fact is that either eq. (5.23) (or eqs. (5.25) and (5.26) without the extra term), and also the relation (5.22), have a possible interpretation in terms of the 't Hooft string picture of the $1/N$ expansion. Thus at least in the case of $L = 1, 2$, eqs. (5.23) and (5.22) still do, so one can hope that there is a correct relation at higher L yet to be determined.

't Hooft's idea was to construct string worldsheets out of Yang-Mills Feynman diagrams by drawing simplified diagrams (depending only on a particle's color structure), with adjoint fields represented by double lines and fundamental fields (quarks) by single lines. In $\mathcal{N} = 4$

[8] The three-loop formula of Sterman and Tejeda-Yeomans was not derived by an explicit calculation, but its $1/\epsilon^3$ term, which gives the leading term in $A^{(3,3)}$, is probably correct. Similarly, we did no explicit three-loop calculation for $\mathcal{N} = 8$ supergravity, and the exponentiation conjecture for the leading IR divergence in ref. [22] was based on the two-loop exponentiation formula and the fact that in the theories where this was studied, at least the leading IR divergences exponentiate. So it is possible, but very unlikely, that eq. (5.23) holds beyond two loops.

SYM, all fields are in the adjoint representation, therefore all 't Hooft diagrams are composed of double lines only. In this picture, an index line loop represents a color trace,[9] contributing a factor of N.

As we noted earlier, the leading-color amplitudes correspond to planar diagrams, carrying a factor of N^L at L loops. A subleading-color contribution down by $1/N$ (e.g., the leading term of the double-trace amplitude $A_{4;3}$) comes from a diagram missing a line loop (= color trace), which for the 't Hooft diagrams of $\mathcal{N}=4$ SYM can only come from twists of the external (on the outside boundary of the corresponding planar diagram) double lines[10] in such a way that index loops of the external gluons become disconnected, giving a diagram with the topology of a hole (annulus). A subleading-color amplitude down by $1/N^2$ corresponds to a diagram which also has twists of internal (not belonging to the outside boundary of the planar diagram) double lines giving a nonplanar diagram with a handle, which does not modify the external color trace. For higher-point functions, we could have multiple $(k+1)$ trace amplitudes down by $1/N^k$, coming from diagrams with the topology of a surface with k holes (open string loops).

't Hooft's proposal for a relation to string theory associated a set of Yang-Mills diagrams with a string worldsheet. (The original idea for the string to live in four flat dimensions never quite worked out in detail, though for $\mathcal{N}=4$ SYM theory, the $AdS_5\times S^5$ string may be the correct construction.) Thus, a planar (leading-color) SYM diagram corresponds to a tree-level string worldsheet, a $1/N$ subleading-color SYM diagram with the topology of a hole (annulus) corresponds to an open-string one-loop worldsheet, and a $1/N^2$ subleading-color SYM diagram with a handle topology corresponds to a closed-string one-loop worldsheet.

It is possible, and our two-loop relation between $A^{(L,L)}_{\rm SYM}$ and $A^{(L)}_{\rm SG}$ for $L=1,2$ seems to suggest this, that one can reduce each closed-string loop to two open-string loops (this relation is certainly valid for vacuum diagrams). The open-string loop comes with a factor of $g^2_{\rm SYM}=g^2_{\rm open}=g_s$, whereas a closed-string diagram comes with a factor of $g^4_{\rm SYM}=g^2_{\rm closed}=g_s^2$, so that one would have

$$\text{one loop closed string} \;=\; (\text{one loop open string})^2 \tag{5.27}$$

reducing each $1/N$ factor in the SYM amplitudes to an open-string loop; thus $A^{(L,k)}$ corresponds to k open-string loops. But now, if we also adopt the rule

$$\text{one loop open 't Hooft string (in 4d!)} = \text{ one loop in } \mathcal{N}=8 \text{ supergravity in 4d} \tag{5.28}$$

then eqs. (5.22) and (5.23) (or eqs. (5.25) and (5.26) with no extra terms) have a simple interpretation: $A^{(L,k)}$ corresponds to k loops in the supergravity expansion.

[9]The g- and N-dependence of an n-gluon amplitude can be written suggestively as $g^n(g^2N)^I(g^2)^{2H+B-2}$, where I=number of index loops, H=number of handles, and B=number of boundaries of the Feynman diagram=worldsheet. Each index loop brings a factor of N, as well as a factor of g^2, and $2H+B-2$ characterizes the topology of the surface. Thus an index loop is associated with the same coupling behaviour as an open string loop (splitting and rejoining).

[10] For example, the one-loop four-gluon diagram contains three possible cases, with two twists adjacent to an external gluon a_1 giving ${\rm Tr}(T^{a_1}){\rm Tr}(T^{a_2}T^{a_3}T^{a_4})$, which vanishes for $SU(N)$, two twists on opposite side of the box giving, for example, ${\rm Tr}(T^{a_1}T^{a_2}){\rm Tr}(T^{a_3}T^{a_4})$, and twists on all four double lines giving ${\rm Tr}(T^{a_1}T^{a_3}){\rm Tr}(T^{a_2}T^{a_4})$.

Using the rule (5.28) for the $k = L$ relation (5.23), for which the number of loops in SYM equals the number of loops in supergravity, there are no internal loops in the 't Hooft double line diagram, thus the diagram is defined exclusively by its handles and holes. Then eq. (5.23) just corresponds to replacing $M_{\mathrm{SYM}}^{(L,L)}$ with $M_{\mathrm{SG}}^{(L)}$, $\sqrt{2}\lambda_{\mathrm{SYM}}$ with $\lambda_{\mathrm{SG}}u$ (the effective supergravity coupling constant) and, since supergravity amplitudes are permutation-symmetric whereas SYM amplitudes are not (the position of the twists breaks the symmetry of the Feynman diagram, even if nothing else does), averaging over cyclic permutations. If the correction term were not present in the square brackets in eqs. (5.25) and (5.26), then we would just need to replace $\lambda_{\mathrm{SYM}}^{2k+1}$ with $(\lambda_{\mathrm{SG}}^2(s^2+t^2+u^2)/3)^k\lambda_{\mathrm{SG}}u$ and $\lambda_{\mathrm{SYM}}^{2k+2}$ with $(\lambda_{\mathrm{SG}}^2(s^2+t^2+u^2)/3)^k(\lambda_{\mathrm{SG}}u)^2$ instead. Since the correction terms are present, the interpretation is not clear.

It is also not clear how to derive the rule (5.28), or why such a rule should even be possible. It is reminiscent of AdS/CFT, but then it is not clear why we get supergravity in 4d and not some higher-dimensional space. If true, it could be a manifestation of a different kind of duality, relating weak coupling with weak coupling, as also advocated in ref. [33].

On the other hand, perhaps the rule (5.28) is only an approximation. When compactifying string theory down to 4d, the supergravity modes could in principle get mixed up with other string modes, and the loop expansion of supergravity combined with other terms, so it is in principle possible that a modification of the rule (5.28) could account for the modified relations (5.25) and (5.26), and make them precise beyond leading order.

6 Transcendentality

One may associate with each term in an operator or amplitude a degree of transcendentality as follows: each factor of ζ_k, π^k, $\log^k z$ (where z is any ratio of momentum invariants), or any polylogarithm of total degree k has transcendentality k and the transcendentality of a product of factors is additive. By uniform transcendentality k_0, we mean that in an ϵ-expansion $\sum_k a_k\epsilon^k$ (for dimensional regularization), a_k has transcendentality $k+k_0$. Maximal transcendentality means that k_0 has a maximal value. In the case of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity, maximal transcendentality means that L-loop amplitudes have uniform transcendentality $k_0 = 2L$; that is, all terms proportional to $(\lambda/\epsilon^2)^L \cdot \epsilon^k$ have degree of transcendentality k.

In this section, we first study the transcendentality of the IR-divergent Catani operators $\mathbf{I}^{(L)}$, which determine all the IR-divergent terms of n-gluon amplitudes. The $\mathcal{N}=4$ Catani operators have uniform transcendentality, and moreover constitute the maximum transcendentality piece of the corresponding QCD operators. We then go on to examine the transcendentality of the subleading-color amplitudes of $\mathcal{N}=4$ SYM theory. These too have uniform transcendentality (as do the leading-color amplitudes), but in this case do not constitute the entire maximum transcendentality piece of the corresponding QCD amplitudes.

6.1 Transcendentality of IR-divergent n-point operators

From eqs. (3.3)-(3.5), one may see that, for the L-loop amplitudes to have maximal transcendentality $k_0 = 2L$, the L-loop Catani operator $\mathbf{I}^{(L)}$ must have uniform transcendentality $k_0 = 2L$. We will show in this subsection that the one- and two-loop Catani operators of $\mathcal{N} = 4$ SYM theory (and the three-loop operator, up to an undetermined term of $\mathcal{O}(1/\epsilon)$) do satisfy this, and moreover constitute the maximal transcendentality piece of the QCD Catani operators. The review [34] is useful for this discussion.

The one-loop Catani operator for QCD is [18]

$$\mathbf{I}^{(1)}(\epsilon) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} \mathbf{T}_i \cdot \mathbf{T}_j \left[\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{i,j}}\right)^{\epsilon} + \frac{\gamma_i}{\mathbf{T}_i^2}\frac{1}{\epsilon}\left(\frac{\mu^2}{-s_{i,j}}\right)^{\epsilon}\right] \tag{6.1}$$

where $\gamma_i = b_0 = \frac{11}{6}N - \frac{2}{3}T_R N_f$ for gluons. The first term in brackets has uniform transcendentality $k_0 = 2$ while the second term in brackets has uniform transcendentality $k_0 = 1$, and hence is subleading in transcendentality. For $\mathcal{N} = 4$ SYM, however, $b_0 = 0$, so in this case $\mathbf{I}^{(1)}(\epsilon)$ has uniform transcendentality $k_0 = 2$, and is given by the maximal transcendentality part of the QCD operator (6.1).

The two-loop Catani operator for QCD is [18]

$$\begin{aligned}\mathbf{I}^{(2)}(\epsilon) &= -\frac{1}{2}\mathbf{I}^{(1)}(\epsilon)\left[\mathbf{I}^{(1)}(\epsilon) + \frac{2b_0}{\epsilon}\right] + c(\epsilon)\left[K + \frac{b_0}{\epsilon}\right]\mathbf{I}^{(1)}(2\epsilon) \\ &\quad + \frac{c(\epsilon)}{4\epsilon}\left[-\sum_{i=1}^{n}\sum_{j\neq i}^{n}\mathbf{T}_i\cdot\mathbf{T}_j \frac{H_i^{(2)}}{\mathbf{T}_i^2}\left(\frac{\mu^2}{-s_{ij}}\right)^{2\epsilon} + \hat{\mathbf{H}}^{(2)}(\epsilon)\right]\end{aligned} \tag{6.2}$$

with $\hat{\mathbf{H}}^{(2)}$ given by [29–31]

$$\hat{\mathbf{H}}^{(2)} = i\sum_{(i_1,i_2,i_3)} f^{a_1a_2a_3}T_{i_1}^{a_1}T_{i_2}^{a_2}T_{i_3}^{a_3}\log\left(\frac{-s_{i_1i_2}}{-s_{i_2i_3}}\right)\log\left(\frac{-s_{i_2i_3}}{-s_{i_3i_1}}\right)\log\left(\frac{-s_{i_3i_1}}{-s_{i_1i_2}}\right). \tag{6.3}$$

Again, in general, $\mathbf{I}^{(2)}$ contains terms of mixed transcendentality, but for $\mathcal{N} = 4$ SYM, one has $b_0 = 0$, $K = -\zeta_2 N$, and $H_i^{(2)} = \frac{1}{2}\zeta_3 N^2$ so only the terms of maximal transcendentality remain. (The expression $c(\epsilon)$ itself is of uniform transcendentality $k_0 = 0$, as can be seen from eq. (3.8).) Thus we see that for $\mathcal{N} = 4$ SYM, the operator $\mathbf{I}^{(2)}$ has maximal uniform transcendentality $k_0 = 4$.

At three loops, we consider the $\mathcal{N} = 4$ SYM Catani operator (3.29). All the terms through $1/\epsilon^2$ have uniform transcendentality $k_0 = 6$, thus maximal. (The $1/\epsilon$ term contains $\mathbf{\Gamma}^{(3)}$, which we have not computed.) Moreover, from eq. (30) of ref. [19], we see that the additional terms in the three-loop QCD Catani operator, proportional to b_0, contain at most terms of uniform transcendentality $k_0 = 5$, and thus subdominant. We expect the same pattern to hold for the three-loop anomalous dimension matrix $\mathbf{\Gamma}^{(3)}$.

Based on these results, it is natural to expect that the L-loop $\mathcal{N} = 4$ SYM Catani operator $\mathbf{I}^{(L)}$ will also be of uniform transcendentality, and the maximal transcendentality piece of the QCD operator.

6.2 Transcendentality of two-loop four-gluon amplitudes

To examine the transcendentality of the $\mathcal{N}=4$ SYM four-gluon amplitudes, one must look at the exact expressions for the planar and non-planar loop integrals. From the explicit Laurent expansions given in refs. [8,20], one may see that, while the one- and two-loop planar integrals (2.8) and (2.12) have uniform transcendentality $k_0 = 2$ and $k_0 = 4$ respectively, the two-loop nonplanar integral (2.15) does not, as it contains terms of subleading transcendentality.

The leading-color one- and two-loop amplitudes $A^{(1,0)}$ and $A^{(2,0)}$, which are built from one- and two-loop planar integrals therefore have uniform transcendentality, as is already well known. The subleading-color one-loop amplitude $A^{(1,1)}$ (and therefore the one-loop supergravity amplitude), which by eq. (2.10) is also built from the one-loop planar integral, also has uniform transcendentality $k_0 = 2$.

It is not obvious that the two-loop subleading-color amplitudes $A^{(2,1)}$ and $A^{(2,2)}$, which are built from two-loop planar and non-planar integrals (cf. eqs. (2.13) and (2.14)) have uniform transcendentality, but we have verified, using the expressions in ref. [22], that all the terms of subdominant transcendentality cancel out, and that the full nonplanar amplitudes have uniform transcendentality $k_0 = 4$, at least through $\mathcal{O}(\epsilon^0)$.

It was previously observed that the two-loop $\mathcal{N}=8$ supergravity amplitude (5.18), which is also built from the two-loop non-planar integral (2.15), nonetheless has uniform transcendentality [22, 23].

Given that the $\mathcal{N}=4$ SYM four-gluon amplitudes (at least through two loops) have uniform transcendentality, the question arises whether they constitute the entire maximum transcendentality piece of the corresponding pure QCD amplitudes [35].

At one loop, the leading-color four-gluon QCD amplitude is [28]

$$M^{\text{gluon}}_{\lambda_1\lambda_2\lambda_3\lambda_4} = (1-\epsilon\delta_R) M^{\text{scalar}}_{\lambda_1\lambda_2\lambda_3\lambda_4} - 4M^{\mathcal{N}=1}_{\lambda_1\lambda_2\lambda_3\lambda_4} + M^{\mathcal{N}=4}_{\lambda_1\lambda_2\lambda_3\lambda_4} \tag{6.4}$$

where λ_i denote helicities. The $\mathcal{N}=4$ SYM four-gluon amplitude $M^{\mathcal{N}=4}_{\lambda_1\lambda_2\lambda_3\lambda_4}$ is nonzero only for helicities $--++$ or $-+-+$, and given by $Box^{(4)}(s,t)$, which has uniform transcendentality, starting with $1/\epsilon^2$ (the ϵ^n term has transcendentality $n+2$). The terms $(1-\epsilon\delta_R)M^{\text{scalar}}_{--++}$ and $M^{\mathcal{N}=1}_{--++}$ are decomposed into $Bub^{(6)}$, $\epsilon Box^{(8)}$, $\epsilon Box^{(6)}$, $Bub^{(4)}$, and terms with lower transcendentality, and from the explicit expressions in ref. [28], we see that they all start at most with $1/\epsilon$, thus at order ϵ^n have transcendentality at most $n+1$. The terms $(1-\epsilon\delta_R)M^{\text{scalar}}_{-+-+}$ and $M^{\mathcal{N}=1}_{-+-+}$, however, contain the finite term $Box^{(6)}$, which has pieces of (maximal) transcendentality two. Hence, only in the case of the one-loop amplitude with helicity $--++$ are the maximal transcendentality terms of QCD given by the $\mathcal{N}=4$ SYM result [36].

The one-loop U(1) decoupling identity (2.9) holds for both $\mathcal{N}=4$ SYM and for $\mathcal{N}=0$ (pure QCD). Since the maximal transcendentality terms of the leading-color one-loop four-gluon QCD amplitude with helicity $--++$ are given by the corresponding $\mathcal{N}=4$ SYM amplitudes, the decoupling identity implies the same result for the subleading-color one-loop amplitudes.

At two loops, this does not hold even for the leading-color amplitude with helicity $--++$. The leading-color two-loop QCD amplitude [28] contains terms of transcendentality two that are not contained in the corresponding $\mathcal{N}=4$ SYM amplitude [36].

7 Conclusions

In this paper we have studied the subleading-color (nonplanar) contributions to the $\mathcal{N}=4$ SYM four-gluon amplitude. Explicit expressions for the IR-divergent terms of the subleading-color amplitudes were computed through three loops using the formalisms of Catani and of Sterman and Tejeda-Yeomans. We extrapolated these results to conjecture the form of the leading IR divergences of the N-independent subleading-color amplitude $A^{(L,L)}$ in eqs. (4.45) and (4.46).

We have also demonstrated some connections between $\mathcal{N}=4$ SYM four-gluon amplitudes and $\mathcal{N}=8$ supergravity four-graviton amplitudes. The one-loop subleading-color SYM amplitude is proportional to the one-loop supergravity amplitude, the proportionality constant being the ratio of the coupling constant $\lambda_{\rm SYM}$ for SYM and the dimensionless effective coupling $\lambda_{\rm SG} u$ for supergravity. Various relations exist between the two-loop subleading-color SYM amplitudes and one- and two-loop supergravity amplitudes, as detailed in sec. 5. The SYM/supergravity connection is most transparent in terms of ratios of loop amplitudes to tree amplitudes $M^{(L)} = A^{(L)}/A^{(0)}$. The relation (5.23) between L-loop SYM and L-loop supergravity amplitudes, which is valid for $L \leq 2$, is understood by replacing $\lambda_{\rm SYM}$ with $\lambda_{\rm SG} u$ and summing over permutations. The simple relation (5.23), however, fails at three loops and beyond (assuming that we have correctly determined the leading divergences of $M_{\rm SYM}^{(L,L)}$ and $M_{\rm SG}^{(L)}$). Instead, we obtain the relations (5.25) and (5.26), which do not have a simple interpretation. If eq. (5.23) were correct (or eqs. (5.25) and (5.26) had no extra terms), we would have had a simple, albeit nonintuitive, interpretation in terms of the 't Hooft picture (equating the topological expansion of SYM Feynman diagrams with string worldsheets). Perhaps a modification of this picture can be found that would relate the subleading-color (nonplanar) $\mathcal{N}=4$ SYM amplitudes to the $\mathcal{N}=8$ supergravity amplitudes to all loop orders.

Our one and two-loop results suggest the possibility of a weak-weak duality between $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity (see also ref. [33]), in contrast to the usual strong-weak AdS/CFT duality. Such a duality, however, would require a relation between SYM and supergravity amplitudes at three loops and beyond, a relation we have failed to find. Since the gauge theory expansion has two parameters, $\lambda_{\rm SYM} = g^2 N$ and $1/N$ (corresponding to α' and g_s in the string picture), whereas the loop expansion of $\mathcal{N}=8$ supergravity has only one parameter, $\lambda_{\rm SG} = (\kappa/2)^2$, it is perhaps unlikely that such a duality could exist without taking into account stringy corrections. It is possible that one needs to consider the mixing of other string theory modes into the loop expansion of supergravity, giving the extra terms in eqs. (5.25) and (5.26).

The one-loop supergravity amplitude appears in many different places in subleading-color SYM amplitudes. Examples include the two leading IR-divergent terms in eqs. (4.26), (4.34), (4.45), and (4.46), as well as the full IR divergence in eqs. (5.13) and (5.15).

Finally, we have examined the issue of transcendentality of the $\mathcal{N}=4$ SYM subleading-color amplitudes and of the Catani operators. Up to two loops, the nonplanar amplitudes have uniform transcendentality, as is already known for the planar amplitudes. The $\mathcal{N}=4$ SYM Catani operators (at least through three loops) also have uniform transcendentality, and constitute the maximum transcendentality piece of the QCD Catani operators.

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